

REDUCTION THEOREM FOR LATTICE COHOMOLOGY

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ABSTRACT. The lattice cohomology of a plumbed 3-manifold M associated with a connected negative definite plumbing graph is an important tool in the study of topological properties of M , and in the comparison of the topological properties with analytic ones when M is realized as complex analytic singularity link. By definition, its computation is based on the (Riemann–Roch) weights of the lattice points of \mathbb{Z}^s , where s is the number of vertices of the plumbing graph. The present article reduces the rank of this lattice to the number of ‘bad’ vertices of the graph. (Usually the geometry/topology of M is codified exactly by these ‘bad’ vertices via surgery or other constructions. Their number measures how far is the plumbing graph from a rational one.)

The effect of the reduction appears also at the level of certain multivariable (topological Poincaré) series as well. Since from these series one can also read the Seiberg–Witten invariants, the reduction theorem provides new formulae for these invariants too. The reduction also implies the vanishing $\mathbb{H}^q = 0$ of the lattice cohomology for $q \geq \nu$, where ν is the number of ‘bad’ vertices. (This bound is sharp.)

1. INTRODUCTION

1.1. Let M be a plumbed 3-manifold given by a connected negative definite plumbing graph, hence, it can be considered as the link of a normal surface singularity as well. In this article we will assume that M is a rational homology sphere.

The second author introduced in [11, 16] a cohomology theory $\mathbb{H}^*(M)$ associated with M , called the *lattice cohomology*. This development was strongly influenced by the *Artin–Laufer program* of normal surface singularities (targeting topological characterization of certain analytic invariants) and by the work of Ozsváth and Szabó on the *Heegaard–Floer theory*, especially [26] (see also their long list of papers in the subject, e.g. [27, 28]).

The lattice cohomology is purely combinatorial. Conjecturally it contains all the information about the Heegaard–Floer homology of M too, cf. [16]. (The conjecture was verified for several families, cf. [11, 24, 29, 31].) Recently Ozsváth, Stipsicz and Szabó in [29] established a spectral sequence starting from the lattice cohomology and converging to the Heegaard–Floer homology. Moreover, they considered the relative version (for knots in M) too [30, 31]. A different version of the relative lattice cohomology associated with local plane curve singularities was identified with the motivic Poincaré series of such germs [7].

Moreover, [17] proves that the normalized Euler characteristic of the lattice cohomology (similarly as of the Heegaard–Floer homology) coincides with the normalized Seiberg–Witten invariant of the link M . This provides a new combinatorial formula for the Seiberg–Witten invariants as well.

From analytic point of view, the ranks of the lattice cohomology modules and their Euler characteristic have subtle connection with certain analytic invariants of analytic realizations

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of M as singularity links [11, 16, 17]. For example, the existence of the nontrivial higher homologies explain conceptually the failure in the pathological cases of the ‘Seiberg–Witten invariant conjecture’, see [20, 21, 22, 23] and [10] for counterexamples. (This conjecture, formulated by Nicolaescu and the second author, predicted for certain analytic structures the coincidence of the geometric genus with the Seiberg–Witten invariant of the link. It is an extension of the Casson Invariant Conjecture of Neumann–Wahl [25]. It was verified for important families of singularities, e.g. in the case of splice quotient singularities [23, 2].) In this sense the lattice cohomology makes a bridge between the analytic and topological/combinatorial invariants of the singularity. For the details the reader is invited to consult [11, 16].

1.2. Usually, the explicit computation of the lattice cohomology is very hard. A priori, it is based on the computation of the weights of all lattice points (of a certain \mathbb{Z}^s) and on the description of those ‘regions’, where the weights are less than a fixed integer. The lattice which appears in the construction has a very ‘large’ rank: it is the number of vertices of the corresponding plumbing/resolution graph G of M . (The weight usually is provided by a Riemann–Roch formula.)

The main result of the present work establishes a *reduction procedure* (Theorem 3.3.3), which reduces the rank of the lattice to ν , the number of ‘bad’ vertices in the plumbing graph G . This number is definitely much smaller (usually it is even smaller than the number of nodes of the graph, e.g. in the case of the star-shaped graphs it is at most one). The number of ‘bad’ vertices provides some kind of ‘filtration’ of negative definite plumbing graphs/manifolds, which measures how far the graph stays from a rational graph. (For more comments and examples see 1.4 too.)

The methods used in the Reduction Theorem and in its proof have their origin in [11, 16], although technically the present situation is more sophisticated. The main ingredient is the generalization of the ‘special’ cycles defined by the second author in [11, 7.6.] for ‘almost rational graphs’ (i.e. when $\nu=1$).

The effects of the reduction appear not only at the level of the cohomology modules. The lattice cohomology has subtle connections with a certain multivariable Poincaré series (defined combinatorially from the graph, but which resonates and sometimes equals, the multivariable Poincaré series associated with the divisorial filtration indexed by all the divisors in the resolution, provided by certain analytic realizations) [15, 17, 19]. For example, the Seiberg–Witten invariant appears as the ‘periodic constant’ of this series [17, 2, 23]. The number of variables of this series is again the number of vertices of the plumbing graph. One of the applications of the Reduction Theorem (and its proof) is that the series obtained from the previous one by eliminating all the variables except those corresponding to the ‘bad’ vertices still contains all these lattice cohomological information.

The reduction recovers several known results as well: e.g. the vanishing of the reduced lattice cohomology for rational graphs, proved in [16, §4]. More generally, it implies the vanishing property $\mathbb{H}^q(M) = 0$ whenever $q \geq \nu$. An alternative proof of this fact can be found in [18], where the proof uses surgery exact sequences. This vanishing is sharp, consider e.g. the connected sum K of ν copies of the $(2, 3)$ –torus knot, and take the $(-d)$ –surgery of the 3–sphere S^3 along K , for some $d \in \mathbb{Z}_{>0}$. Then the minimal number of bad vertices is ν , and $\mathbb{H}^{\nu-1}(S^3_{-d}(K)) = \mathbb{Z}$ [24].

1.3. The organization of the note is the following: section 2 contains some generalities about the plumbing graphs and reviews the construction and different interpretations of the lattice cohomology. The next section defines the ‘special’ cycles $x(\mathbf{i})$, the family of ‘bad’ vertices and provides several technical preliminary results about the *generalized Laufer computation sequences*. (For the original sequences introduced by Laufer, see [8, 9].) At the

end of this section we state the Reduction Theorem 3.3.3. The proof is given in section 4. It starts with several simplification steps. The ‘original’ and ‘reduced’ cohomology groups are compared by a projection, in which by a Leray type argument the isomorphism is guaranteed by the fact that all the fibers are non-empty and contractible. Even the proof of the non-emptiness is rather hard. The contraction is done in several steps, and is guided by high generalizations of properties of computation sequences. The last section contains the corresponding consequences regarding the Poincaré series and their connection with the Seiberg–Witten invariants.

1.4. We wish to emphasize that the reduction to ‘bad’ vertices is not just a technical procedure. Usually, the geometry of the singularity link, or the key information about the structure of the 3-manifold, is coded by these vertices. In other words, by a good choice of the bad vertices we connect in a direct way the structure of the lattice cohomology with the essential geometrical/topological structure of M .

Let us exemplify this statement by some examples.

First we define the set of bad vertices. A graph has no bad vertices if it is rational. Otherwise, if one has to decrease the Euler number of (at most) ν vertices to get a rational graph, we say that these vertices are the ‘bad’ ones (they obstruct the graph to be rational).

For example, a minimal good star-shaped graph has at most one bad vertex, namely the central one. In this case, the sequence $x(i)$ ($i \in \mathbb{Z}_{\geq 0}$) and the weights of its terms are closely related with Pinkham’s computation ([32]) of the geometric genus and of the Poincaré series of weighted homogeneous singularities, see e.g. [21, 11]. In fact, the Poincaré series associated with the analytic \mathbb{C}^* -action coincides with the above discussed combinatorial series associated with the central vertex. (As a consequence, the geometric genus coincides with the normalized Seiberg–Witten invariant of the link.)

Another example: let K be the connected sum of ν irreducible algebraic knots $\{K_i\}_{i=1}^\nu$ of S^3 . Consider the surgery 3-manifold $M = S^3_{-d}(K)$ ($d \in \mathbb{Z}_{\geq 0}$). Then the minimal number of bad vertices is exactly ν , and they can be related with the knots, e.g., the lattice cohomology associated with these vertices is guided by the semigroups of the knot components K_i (for details see [24], where the Reduction Theorem already was applied).

Even the ‘naive case of all nodes’ can be interesting in the right situation. If the graph is minimal good, then reducing the Euler numbers of nodes we get a minimal rational graph (with reduced fundamental cycle), hence the set of all nodes might serve as set of bad vertices. This becomes especially meaningful when we consider e.g. the graph/link of a hypersurface singularity with non-degenerate Newton principal part. In this case the nodes correspond to the faces of the Newton diagram (by toric resolution). Hence, this choice of the bad vertices establishes the connection with the combinatorics of the source object, the Newton diagram.

Although the present article contains basically ‘only’ the theoretical statements, different (new) applications with detailed computations will follow it in forthcoming notes.

2. REVIEW OF THE LATTICE COHOMOLOGY

2.1. Generalities about plumbing graphs. We consider a connected negative definite plumbing graph G . It can be realized as the resolution graph of some normal surface singularity $(X, 0)$, and the link M of $(X, 0)$ can be considered as the plumbed 3-manifold associated with G . In the sequel we assume that M is a rational homology sphere, or, equivalently, G is a tree and all the genera decorations are zero. For more details regarding this section, see e.g. [11, 12, 14, 16].

Let \tilde{X} be the smooth 4-manifold with boundary M obtained either by plumbing disc bundles along G , or via a resolution $\pi : \tilde{X} \rightarrow X$ of $(X, 0)$ with resolution graph G . Then

$L = H_2(\tilde{X}, \mathbb{Z})$ is generated by $\{E_j\}_{j \in \mathcal{J}}$, the cores of the plumbing construction (or the irreducible components of the exceptional divisor $E := \pi^{-1}(0)$ of π). L is a lattice via the negative definite intersection form $\mathfrak{I} := \{(E_j, E_i)\}_{j,i}$. Let L' be the dual lattice $\{l' \in L \otimes \mathbb{Q} : (l', L) \subseteq \mathbb{Z}\}$. L' is generated by the (anti)dual elements E_j^* defined via $(E_j^*, E_k) = -\delta_{jk}$ (the negative of the Kronecker symbol). Set $H := L'/L$. Then $H_1(M, \mathbb{Z}) = H$. Clearly, the E_j^* are the columns of $-\mathfrak{I}^{-1}$, and is known that

(2.1.1) all the entries of E_j^* are strict positive.

If $l'_k = \sum_j l'_{kj} E_j$ for $k = 1, 2$, then we write $\min\{l'_1, l'_2\} := \sum_j \min\{l'_{1j}, l'_{2j}\} E_j$, and $l'_1 \leq l'_2$ if $l'_{1j} \leq l'_{2j}$ for all $j \in \mathcal{J}$. Furthermore, if $l' = \sum_j l'_j E_j$ then we set $|l'| := \{j \in \mathcal{J} : l'_j \neq 0\}$ for the *support* of l' .

The set of *characteristic elements* are defined as

$$Char := \{k \in L' : (k, x) + (x, x) \in 2\mathbb{Z} \text{ for any } x \in L\}.$$

The unique rational cycle $k_{can} \in L'$ which satisfies the system of *adjunction relations* $(k_{can}, E_j) = -(E_j, E_j) - 2$ for all j is called the *canonical cycle*. Then $Char = k_{can} + 2L'$. There is a natural action of L on $Char$ given by $l * k := k + 2l$; its orbits are of type $k + 2L$. Obviously, H acts freely and transitively on the set of orbits by $[l'] * (k + 2L) := k + 2l' + 2L$.

The first Chern class realizes an identification between the $spin^c$ -structures $Spin^c(\tilde{X})$ on \tilde{X} and $Char \subseteq L'$. $Spin^c(\tilde{X})$ is an L' torsor compatible with the above action of L' on $Char$.

All the $spin^c$ -structures on M are obtained by restriction $Spin^c(\tilde{X}) \rightarrow Spin^c(M)$, $Spin^c(M)$ is an H torsor, and the actions are compatible with the factorization $L' \rightarrow H$. Hence, one has an identification of $Spin^c(M)$ with the set of L -orbits of $Char$, and this identification is compatible with the action of H on both sets. In this way, any $spin^c$ -structure of M will be represented by an orbit $[k] := k + 2L \subseteq Char$ (see [6]).

The *canonical* $spin^c$ -structure corresponds to $[-k_{can}]$.

2.2. $\mathbb{Z}[U]$ -modules. The lattice cohomology has a graded $\mathbb{Z}[U]$ -module structure. One of its building blocks, \mathcal{T}_m^+ , is defined as follows, cf. [26, 11].

Consider the graded $\mathbb{Z}[U]$ -module $\mathbb{Z}[U, U^{-1}]$, and denote by \mathcal{T}_0^+ its quotient by the submodule $U \cdot \mathbb{Z}[U]$. Its grading is given by $\deg(U^{-d}) = 2d$ ($d \geq 0$). Next, for any graded $\mathbb{Z}[U]$ -module P with d -homogeneous elements P_d , and for any $r \in \mathbb{Q}$, we denote by $P[r]$ the same module graded (by \mathbb{Q}) in such a way that $P[r]_{d+r} = P_d$. Then set $\mathcal{T}_r^+ := \mathcal{T}_0^+[r]$.

2.3. Lattice cohomology associated with \mathbb{Z}^s and a system of weights [16]. We fix a free \mathbb{Z} -module, with a fixed basis $\{E_j\}_{j=1}^s$, denoted by \mathbb{Z}^s . It is also convenient to fix a total ordering of the index set \mathcal{J} , which in the sequel will be denoted by $\{1, \dots, s\}$. Using the pair $(\mathbb{Z}^s, \{E_j\}_j)$ and a system of weights we determine a cochain complex whose cohomology is our central object.

2.3.1. The cochain complex. $\mathbb{Z}^s \otimes \mathbb{R}$ has a natural cellular decomposition into cubes. The set of zero-dimensional cubes is provided by the lattice points \mathbb{Z}^s . Any $l \in \mathbb{Z}^s$ and subset $I \subseteq \mathcal{J}$ of cardinality q define a q -dimensional cube, denoted by (l, I) (or only by \square_q) which has its vertices in the lattice points $(l + \sum_{j \in I'} E_j)_{I'}$, where I' runs over all subsets of I . On each such cube we fix an orientation. This can be determined, e.g., by the order $(E_{j_1}, \dots, E_{j_q})$, where $j_1 < \dots < j_q$, of the involved base elements $\{E_j\}_{j \in I}$. The set of oriented q -dimensional cubes defined in this way is denoted by \mathcal{Q}_q ($0 \leq q \leq s$).

Let \mathcal{C}_q be the free \mathbb{Z} -module generated by oriented cubes $\square_q \in \mathcal{Q}_q$. Clearly, for each $\square_q \in \mathcal{Q}_q$, the oriented boundary $\partial \square_q$ has the form $\sum_k \varepsilon_k \square_{q-1}^k$ for some $\varepsilon_k \in \{-1, +1\}$, where the $(q-1)$ -cubes $\{\square_{q-1}^k\}_k$ are the *oriented faces* of \square_q . Clearly $\partial \circ \partial = 0$, and

the homology of the chain complex $(\mathcal{C}_*, \partial)$ is just the homology of \mathbb{R}^s . A more interesting (co)homology is obtained via a set of *weight functions*.

2.3.2. Definition. A set of functions $w_q : \mathcal{Q}_q \rightarrow \mathbb{Z}$ ($0 \leq q \leq s$) is called a *set of compatible weight functions* if the following hold:

- (a) for any integer $k \in \mathbb{Z}$, the set $w_0^{-1}((-\infty, k])$ is finite;
- (b) for any $\square_q \in \mathcal{Q}_q$ and for any of its faces $\square_{q-1} \in \mathcal{Q}_{q-1}$ one has $w_q(\square_q) \geq w_{q-1}(\square_{q-1})$.

Example 2.3.3. Assume that some $w_0 : \mathcal{Q}_0 \rightarrow \mathbb{Z}$ satisfies (a) for all $k \in \mathbb{Z}$. For any $q \geq 1$ set

$$w_q(\square_q) := \max\{w_0(v) : v \text{ is a vertex of } \square_q\}.$$

Then $\{w_q\}_q$ is a set of compatible weight functions.

In the presence of a set of compatible weight functions $\{w_q\}_q$, one sets $\mathcal{F}^q := \text{Hom}_{\mathbb{Z}}(\mathcal{C}_q, \mathcal{T}_0^+)$. Then \mathcal{F}^q is a $\mathbb{Z}[U]$ -module by $(p * \phi)(\square_q) := p(\phi(\square_q))$ ($p \in \mathbb{Z}[U]$), and it has a \mathbb{Z} -grading: $\phi \in \mathcal{F}^q$ is homogeneous of degree $d \in \mathbb{Z}$ if for each $\square_q \in \mathcal{Q}_q$ with $\phi(\square_q) \neq 0$, $\phi(\square_q)$ is a homogeneous element of \mathcal{T}_0^+ of degree $d - 2 \cdot w(\square_q)$. (In the sequel sometimes we will omit the index q of w_q .)

Next, one defines $\delta_w : \mathcal{F}^q \rightarrow \mathcal{F}^{q+1}$. For this, fix $\phi \in \mathcal{F}^q$ and we show how $\delta_w \phi$ acts on a cube $\square_{q+1} \in \mathcal{Q}_{q+1}$. First write $\partial \square_{q+1} = \sum_k \varepsilon_k \square_q^k$, then set

$$(\delta_w \phi)(\square_{q+1}) := \sum_k \varepsilon_k U^{w(\square_{q+1}) - w(\square_q^k)} \phi(\square_q^k).$$

Then $\delta_w \circ \delta_w = 0$, hence $(\mathcal{F}^*, \delta_w)$ is a cochain complex. Moreover, $(\mathcal{F}^*, \delta_w)$ has an augmentation. Set $\mathbf{m}_w := \min_{l \in \mathbb{Z}^s} w_0(l)$ and choose $l_w \in \mathbb{Z}^s$ such that $w_0(l_w) = \mathbf{m}_w$. Then one defines the $\mathbb{Z}[U]$ -linear map $\epsilon_w : \mathcal{T}_{2\mathbf{m}_w}^+ \rightarrow \mathcal{F}^0$ such that $\epsilon_w(U^{-\mathbf{m}_w - n})(l)$ is the class of $U^{-\mathbf{m}_w + w_0(l) - n}$ in \mathcal{T}_0^+ for any $n \in \mathbb{Z}_{\geq 0}$. Then, ϵ_w is injective, and $\delta_w \circ \epsilon_w = 0$.

2.3.4. Definitions. The homology of the cochain complex $(\mathcal{F}^*, \delta_w)$ is called the *lattice cohomology* of the pair (\mathbb{R}^s, w) , and it is denoted by $\mathbb{H}^*(\mathbb{R}^s, w)$. The homology of the augmented cochain complex

$$0 \longrightarrow \mathcal{T}_{2\mathbf{m}_w}^+ \xrightarrow{\epsilon_w} \mathcal{F}^0 \xrightarrow{\delta_w} \mathcal{F}^1 \xrightarrow{\delta_w} \dots$$

is called the *reduced lattice cohomology* of the pair (\mathbb{R}^s, w) , and it is denoted by $\mathbb{H}_{red}^*(\mathbb{R}^s, w)$. For any $q \geq 0$, both \mathbb{H}^q and \mathbb{H}_{red}^q admit an induced graded $\mathbb{Z}[U]$ -module structure, and one has graded $\mathbb{Z}[U]$ -module isomorphisms $\mathbb{H}^q = \mathbb{H}_{red}^q$ for $q > 0$ and $\mathbb{H}^0 = \mathcal{T}_{2\mathbf{m}_w}^+ \oplus \mathbb{H}_{red}^0$.

2.3.5. Modification. Clearly, instead of all the cubes of \mathbb{R}^s we can consider only those ones which sit in $[0, \infty)^s$, or only in the ‘rectangle’ $R := [0, T_1] \times \dots \times [0, T_s]$ (for some $T_i \in \mathbb{Z}_{\geq 0}$). In such a case, we write $\mathbb{H}^*([0, \infty)^s, w)$ or $\mathbb{H}^*(R, w)$ for the corresponding lattice cohomologies.

2.4. The \mathbb{S}^* -realization. A more geometric realization of the modules \mathbb{H}^* is the following. For each $N \in \mathbb{Z}$, define $S_N = S_N(w) \subseteq \mathbb{R}^s$ as the union of all the cubes \square_q (of any dimension) with $w(\square_q) \leq N$. Clearly, $S_N = \emptyset$, whenever $N < \mathbf{m}_w$. For any $q \geq 0$, set

$$\mathbb{S}^q(\mathbb{R}^s, w) := \oplus_{N \geq \mathbf{m}_w} H^q(S_N, \mathbb{Z}).$$

Then \mathbb{S}^q is $2\mathbb{Z}$ -graded, the $d = 2N$ -homogeneous elements \mathbb{S}_d^q consists of $H^q(S_N, \mathbb{Z})$. Also, \mathbb{S}^q is a $\mathbb{Z}[U]$ -module. The U -action is given by the restriction map $r_{N+1} : H^q(S_{N+1}, \mathbb{Z}) \rightarrow H^q(S_N, \mathbb{Z})$, namely, the N^{th} -component of $U * (\{\alpha_N\}_N)$ is $r_{N+1} \alpha_{N+1}$. Moreover, for $q = 0$, a fixed base-point $l_w \in S_{\mathbf{m}_w}$ provides an augmentation $H^0(S_N, \mathbb{Z}) = \mathbb{Z} \oplus \tilde{H}^0(S_N, \mathbb{Z})$, hence an augmentation of the graded $\mathbb{Z}[U]$ -modules

$$\mathbb{S}^0 = (\oplus_{N \geq \mathbf{m}_w} \mathbb{Z}) \oplus (\oplus_{N \geq \mathbf{m}_w} \tilde{H}^0(S_N, \mathbb{Z})) = \mathcal{T}_{2\mathbf{m}_w}^+ \oplus \mathbb{S}_{red}^0.$$

Theorem 2.4.1. [16] *There exists a graded $\mathbb{Z}[U]$ -module isomorphism, compatible with the augmentations, between $\mathbb{H}^*(\mathbb{R}^s, w)$ and $\mathbb{S}^*(\mathbb{R}^s, w)$. Similar statement is valid for $\mathbb{H}^*([0, \infty)^s, w)$, or for $\mathbb{H}^*(\prod_i [0, T_i], w)$ too.*

From now on we denote both realizations with the same symbol \mathbb{H}^* , no matter which one we use.

2.5. The lattice cohomology associated with a plumbing graph. Let G be a negative definite plumbing graph as in 2.1. Let s be the number of vertices. Then we can associate to $L = \mathbb{Z}^s$ the free \mathbb{Z} -module \mathcal{C}_q generated by oriented cubes $\square_q \in \mathcal{Q}_q$, as in 2.3.1.

To any $k \in \text{Char}$ we associate weight functions $\{w_q\}_q$ as follows. First, we define $\chi_k : L \rightarrow \mathbb{Z}$ by

$$\chi_k(l) = -(l, l + k)/2;$$

and we also write $\mathbf{m}_k := \min \{ \chi_k(l) : l \in L \}$. Then the weight functions are defined as in 2.3.3 via $w_0 := \chi_k$. The associated lattice cohomologies will be denoted by $\mathbb{H}^*(G, k)$ and $\mathbb{H}_{red}^*(G, k)$.

It is proved in [16] that $\mathbb{H}_{red}^*(G, k)$ is finitely generated over \mathbb{Z} .

Remark 2.5.1. Although each k provides a different cohomology module, there are only $|L'/L|$ essentially different ones. Indeed, assume that $[k] = [k']$, hence $k' = k + 2l$ for some $l \in L$. Then

$$(2.5.2) \quad \chi_{k'}(x - l) = \chi_k(x) - \chi_k(l) \quad \text{for any } x \in L.$$

Therefore, the transformation $x \mapsto x' := x - l$ realizes the following identification:

$$\mathbb{H}^*(G, k') = \mathbb{H}^*(G, k)[-2\chi_k(l)].$$

2.6. The distinguished representatives k_r . We fix a $spin^c$ -structure $[k]$. Recall, see 2.1, that $[k]$ has the form $k_{can} + 2(l' + L)$ for some $l' \in L'$. Among all the characteristic elements in $[k]$ we will choose a very special one. Consider the (Lipman, or anti-nef) cone

$$\mathcal{S}' := \{l' \in L' : (l', E_v) \leq 0 \text{ for any vertex } v\}.$$

2.6.1. Definition. [11, (5.4–5.5)] We denote by $l'_{[k]} \in L'$ the unique minimal element of $(l' + L) \cap \mathcal{S}'$ and we call $k_r := k_{can} + 2l'_{[k]}$ the *distinguished representative* of the class $[k]$.

For example, since the minimal element of $L \cap \mathcal{S}'$ is the zero cycle, we get $l'_{[k_{can}]} = 0$, and the distinguished representative in $[k_{can}]$ is the canonical cycle k_{can} itself. In general, $l'_{[k]} \geq 0$.

The classes k_r generalize the canonical cycle for different $spin^c$ -structures. Their importance will be transparent below, see also [11, 16, 13] for different applications. The next properties are proved in [16]:

Lemma 2.6.2. (a) $\mathbb{H}^*(G, k_r) \cong \mathbb{H}^*([0, \infty)^s, k_r)$ for any k_r .

(b) The set $\{\mathbb{H}^*(G, k_r)\}_{[k_r]}$ is independent on the plumbing representation G of the 3-manifold M , hence it associates a $\mathbb{Z}[U]$ -module to any pair (M, k_r) , where $[k_r] \in \text{Spin}^c(M)$.

3. THE LATTICE REDUCTION

3.1. Computation sequences. The goal of the present section is to show that the lattice cohomology of the lattice L (or any rectangle of it) can be reduced to a considerably ‘smaller rank object’. The main tool in this reduction is the ‘*theory of computation sequences*’, initiated by Laufer [8, 9]. In this subsection we introduce the needed generalization and we state the main theorem. The idea of the reduction theorem is present already in [11]. The new lattice of rank ν will be associated with a set of ‘bad’ vertices, the new lattice points

are associated with some important cycles of L as distinguished members of Laufer-type computational sequences of L . We start with their definition.

3.1.1. The definition of the lattice points $x(i_1, \dots, i_\nu)$. Suppose we have a family of *distinguished* vertices $\overline{\mathcal{J}} := \{j_k\}_{k=1}^\nu \subseteq \mathcal{J}$ (usually they are defined by some geometric property). Then split the set of vertices \mathcal{J} into the disjoint union $\overline{\mathcal{J}} \sqcup \mathcal{J}^*$. Furthermore, let $\{m_j(x)\}_j$ denote the coefficients of a rational cycle x , that is $x = \sum_{j \in \mathcal{J}} m_j(x) E_j$.

In order to simplify the notation we set $\mathbf{i} := (i_1, \dots, i_j, \dots, i_\nu) \in \mathbb{Z}^\nu$; for any $j \in \overline{\mathcal{J}}$ we write $1_j \in \mathbb{Z}^\nu$ for the vector with all entries zero except at place j where it is 1, and for any $I \subseteq \overline{\mathcal{J}}$ we define $1_I = \sum_{j \in I} 1_j$. Similarly, for any $I \subseteq \mathcal{J}$ set $E_I = \sum_{j \in I} E_j$.

Then the cycles $x(\mathbf{i}) = x(i_1, \dots, i_\nu)$ are defined via the next Proposition.

Proposition 3.1.2. *Fix $[k]$ and $\overline{\mathcal{J}} \subseteq \mathcal{J}$ as above. For any $\mathbf{i} \in (\mathbb{Z}_{\geq 0})^\nu$ there exists a unique cycle $x(\mathbf{i}) \in L$ satisfying the following properties:*

- (a) $m_j(x(\mathbf{i})) = i_j$ for any distinguished vertex $j \in \overline{\mathcal{J}}$;
- (b) $(x(\mathbf{i}) + l'_{[k]}, E_j) \leq 0$ for every ‘non-distinguished vertex’ $j \in \mathcal{J}^*$;
- (c) $x(\mathbf{i})$ is minimal with the two previous properties.

Moreover, (i) $x(0, \dots, 0) = 0$; (ii) $x(\mathbf{i}) \geq 0$; and (iii) $x(\mathbf{i}) + E_{\overline{I}} \leq x(\mathbf{i} + 1_{\overline{I}})$ for any $\overline{I} \subseteq \overline{\mathcal{J}}$.

Proof. The proof is similar to the proof of [11, Lemma 7.6], valid for $\nu = 1$. First we verify the existence of an element $x \in L$ with (a)–(b). By (the proof of) [11, 7.3] there exists $\tilde{x} \geq \sum_{j \in \overline{\mathcal{J}}} E_j$ such that $(\tilde{x} + l'_{[k]}, E_j) \leq 0$ for any $j \in \mathcal{J}$. Take some $a \in \mathbb{Z}_{>0}$ sufficiently large so that $(a-1)l'_{[k]} \in L$, and $h_j := m_j(a\tilde{x} + (a-1)l'_{[k]}) - i_j \geq 0$ for any $j \in \overline{\mathcal{J}}$. Since $l'_{[k]} \geq 0$, this is possible. Then set $x := a\tilde{x} + (a-1)l'_{[k]} - \sum_{j \in \overline{\mathcal{J}}} h_j E_j$. Clearly $m_j(x) = i_j$ for any $j \in \overline{\mathcal{J}}$ and $(x + l'_{[k]}, E_i) = a(\tilde{x} + l'_{[k]}, E_i) - \sum_{j \in \overline{\mathcal{J}}} h_j (E_j, E_i) \leq 0$ for any $i \in \mathcal{J}^*$.

Next, we verify that there is a unique minimal element with (a)–(b). This follows from the fact that if x_1 and x_2 satisfy (a)–(b), then $x := \min\{x_1, x_2\}$ does too. Indeed, for any $j \in \mathcal{J}^*$, at least for one index $n \in \{1, 2\}$ one has $E_j \notin |x_n - x|$. Then $(x + l'_{[k]}, E_j) = (x_n + l'_{[k]}, E_j) - (x_n - x, E_j) \leq 0$.

Finally, we verify (i)–(ii)–(iii). For (ii) write $x(\mathbf{i})$ as $x_1 - x_2$ with $x_1 \geq 0$, $x_2 \geq 0$, $|x_1| \cap |x_2| = \emptyset$. Fix an index $j \in \mathcal{J}^*$. If $j \notin |x_1|$ then $(l'_{[k]} - x_2, E_j) \leq (l'_{[k]} - x_2 + x_1, E_j) \leq 0$. If $j \in |x_1|$ then $(l'_{[k]} - x_2, E_j) \leq (l'_{[k]}, E_j) \leq 0$, cf. 2.6.1. Hence $l'_{[k]} - x_2 \in (l'_{[k]} + L) \cap \mathcal{S}'$, which implies $x_2 = 0$ by the minimality of $l'_{[k]}$. This ends (ii) and shows (i) too. For (iii) notice that $(x(\mathbf{i} + 1_{\overline{I}}) + l'_{[k]}, E_j) - (E_{\overline{I}}, E_j) \leq 0$ for any $j \in \mathcal{J}^*$, hence the result follows from the minimality property (c) applied for $x(\mathbf{i})$. \square

For a precise form of the cycles $x(\mathbf{i}) = x(i_1, \dots, i_\nu) \in L$ in terms of the intersection form, see 5.3.2. These cycles satisfy the following universal property as well.

Lemma 3.1.3. *Fix some $\mathbf{i} \in (\mathbb{Z}_{\geq 0})^\nu$. Assume that $x \in L$ satisfies $m_j(x) = m_j(x(\mathbf{i}))$ for all $j \in \overline{\mathcal{J}}$.*

If $x \leq x(\mathbf{i})$, then there is a ‘generalized Laufer computation sequence’ connecting x with $x(\mathbf{i})$. More precisely, one constructs a sequence $\{x_n\}_{n=0}^t$ as follows. Set $x_0 = x$. Assume that x_n is already constructed. If for some $j \in \mathcal{J}^$ one has $(x_n + l'_{[k]}, E_j) > 0$ then take $x_{n+1} = x_n + E_{j(n)}$, where $j(n)$ is such an index. If x_n satisfies 3.1.2(b), then stop and set $t = n$. Then this procedure stops after finite steps and x_t is exactly $x(\mathbf{i})$.*

Moreover, along the computation sequence $\chi_{k_r}(x_{n+1}) \leq \chi_{k_r}(x_n)$ for any $0 \leq n < t$.

Proof. We show by induction that $x_n \leq x(\mathbf{i})$ for any $0 \leq n \leq t$; then the minimality property (c) of $x(\mathbf{i})$ will finish the argument. For $n = 0$ this is clear. Assume it is true for

x_n . Then we have to verify that $m_{j(n)}(x_n) < m_{j(n)}(x(\mathbf{i}))$. Suppose that this is not true, that is $m_{j(n)}(x(\mathbf{i}) - x_n) = 0$. Then $(x_n + l'_{[k]}, E_{j(n)}) = (x(\mathbf{i}) + l'_{[k]}, E_{j(n)}) - (x(\mathbf{i}) - x_n, E_{j(n)}) \leq 0$, a contradiction.

Finally, notice that $(x_n + l'_{[k]}, E_{j(n)}) > 0$ implies $\chi_{k_r}(x_{n+1}) \leq \chi_{k_r}(x_n)$. \square

Note that the generalized computation sequence usually is not unique, one can make several choices for $j(n)$ at each step n .

If the choice of the *distinguished vertices* $\overline{\mathcal{J}}$ is guided by some specific geometric feature, then the cycles $x(\mathbf{i})$ will inherit further properties (see next subsection).

3.2. Graphs with ‘bad’ vertices. In [16] is proved that the reduced lattice cohomologies of *rational graphs* (see the definition below) are trivial; in particular the lattice cohomology measures how ‘non-rational’ the graph is. Any graph can be transformed into a rational graph by decreasing the decorations of the vertices. Indeed, if all the Euler decorations of a graph G are sufficiently negative (e.g. $(E_j, E) \leq 0$ for any j), then G is rational. This motivates the next definitions.

Recall that a normal surface singularity is rational if its geometric genus is zero. This vanishing property was characterized combinatorially by Artin in terms of the graph [1]:

$$(3.2.1) \quad \text{rationality} \iff \chi_{can}(l) > 0 \text{ for any } l > 0, l \in L.$$

Definition 3.2.2. A connected negative definite graph is *rational* if it is the resolution graph of a rational singularity, that is, if it satisfies Artin’s criterion (3.2.1).

We say that a graph *has ν bad vertices* if one can find a subset of vertices $\{j_k\}_{k=1}^\nu$, called *bad vertices*, such that replacing their decorations $e_j := (E_j, E_j)$ by some more negative integers $e'_j \leq e_j$ we get a rational graph, cf. [11, 16, 18, 26].

In the sequel we fix a (non-necessarily minimal) set $\overline{\mathcal{J}}$ of bad vertices (that is, by modification of their decorations one gets a rational graph). Next, we start to list some additional properties satisfied by the cycles $x(\mathbf{i})$ associated with $\overline{\mathcal{J}}$. The first is an addendum of Lemma 3.1.3.

Lemma 3.2.3. *Fix some $\mathbf{i} \in (\mathbb{Z}_{\geq 0})^\nu$. Assume that $x \in L$ satisfies $m_j(x) = m_j(x(\mathbf{i}))$ for all $j \in \overline{\mathcal{J}}$. Then $\chi_{k_r}(x) \geq \chi_{k_r}(x(\mathbf{i}))$.*

Proof. Write $x = x(\mathbf{i}) - y_1 + y_2$ with $y_1 \geq 0, y_2 \geq 0$, both y_i supported on \mathcal{J}^* , and $|y_1| \cap |y_2| = \emptyset$. Then $\chi_{k_r}(x) = \chi_{k_r}(x(\mathbf{i}) - y_1) + \chi_{k_r}(y_2) + (y_1, y_2) - (x(\mathbf{i}) + l'_{[k]}, y_2) \geq \chi_{k_r}(x(\mathbf{i}) - y_1)$. Indeed, $(y_1, y_2) \geq 0$ by support-argument, $-(x(\mathbf{i}) + l'_{[k]}, y_2) \geq 0$ by definition of $x(\mathbf{i})$, and $\chi_{k_r}(y_2) \geq 0$ since y_2 is supported on a rational subgraph (cf. [11, (6.3)]). On the other hand, by 3.1.3, $\chi_{k_r}(x(\mathbf{i}) - y_1) \geq \chi_{k_r}(x(\mathbf{i}))$. \square

The computation sequence of Lemma 3.1.3 is a generalization of Laufer’s computation sequence targeting Artin’s fundamental cycle z_{min} , the minimal non-zero cycle of $\mathcal{S}' \cap L$ [8]. In fact, for rational graphs, the algorithm is more precise. For further references we cite it here:

3.2.4. Laufer’s Criterion [8]. *Let $\{z_n\}_{n=0}^T$ be the computation sequence (similar as above with $[k] = [k_{can}]$) connecting $z_0 = E_j$ (for some $j \in \overline{\mathcal{J}}$) and the Artin’s fundamental cycle $z_T = z_{min}$. (This means that $z_{n+1} = z_n + E_{j(n)}$ for some $j(n)$, where $(z_n, E_{j(n)}) > 0$.) Then the graph is rational if and only if at every step $0 \leq n < T$ one has $(E_{j(n)}, z_n) = 1$.*

The same statement is true for a sequence connecting $z_0 = E_I$ with z_{min} for any connected E_I .

(Both statement can be reinterpreted by the identity $\chi_{can}(E_I) = \chi_{can}(z_{min}) = 1$.)

In some of the applications regarding the cycles $x(\mathbf{i})$ we do not really need their precise forms, rather the values $\chi_{k_r}(x(\mathbf{i}))$. These can be computed inductively thanks to the following.

Proposition 3.2.5. *For any $k_r \in \text{Char}$, $\mathbf{i} \in (\mathbb{Z}_{\geq 0})^\nu$ and $j \in \overline{\mathcal{J}}$ one has*

$$\chi_{k_r}(x(\mathbf{i} + 1_j)) = \chi_{k_r}(x(\mathbf{i})) + 1 - (x(\mathbf{i}) + l'_{[k]}, E_j).$$

Moreover, $\chi_{k_r}(x(0, \dots, 0)) = 0$.

Proof. We consider the computation sequence $\{x_n\}_{n=0}^t$ connecting $x(\mathbf{i}) + E_j$ and $x(\mathbf{i} + 1_j)$ and we prove that $(x_n + l'_{[k]}, E_{j(n)})$ is exactly 1 for any $0 \leq n < t$. Indeed, we take $z_n := x_n - x(\mathbf{i})$ for $0 \leq n \leq t$ and one verifies that $\{z_n\}_{n=0}^t$ is the beginning of a Laufer sequence $\{z_n\}_{n=0}^T$ (with $t \leq T$) connecting E_j with z_{\min} (as in 3.2.4). Moreover, the values $(z_n, E_{j(n)})$ will stay unmodified for every n if we replace our graph G with the rational graph \tilde{G} by decreasing the decorations of the bad vertices. Therefore, by Laufer's Criterion, $(z_n, E_{j(n)}) = 1$ in \tilde{G} , and consequently in G too. This shows that

$$1 = (x_n - x(\mathbf{i}), E_{j(n)}) = (x_n + l'_{[k]}, E_{j(n)}) - (x(\mathbf{i}) + l'_{[k]}, E_{j(n)}) \geq (x_n + l'_{[k]}, E_{j(n)}).$$

Since $(x_n + l'_{[k]}, E_{j(n)}) > 0$, this number must equal 1.

This shows $\chi_{k_r}(x_{n+1}) = \chi_{k_r}(x_n)$, or $\chi_{k_r}(x(\mathbf{i} + 1_j)) = \chi_{k_r}(x(\mathbf{i}) + E_j)$. \square

The next technical result about computation sequences is crucial in the proof of the main result.

Proposition 3.2.6. *Fix $\mathbf{i} \in (\mathbb{Z}_{\geq 0})^\nu$ and a subset $\overline{\mathcal{J}} \subseteq \overline{\mathcal{J}}$. Let $\mathbf{s}(\mathbf{i}, \overline{\mathcal{J}}) \subseteq \mathcal{J}^*$ be the support of $x(\mathbf{i} + 1_{\overline{\mathcal{J}}}) - x(\mathbf{i}) - E_{\overline{\mathcal{J}}}$.*

(I) *For any support $\mathbf{s}' \subseteq \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})$ one can find a generalized Laufer computation sequence $\{x_n\}_{n=0}^t$ as in Lemma 3.1.3 connecting $x_0 = x(\mathbf{i}) + E_{\overline{\mathcal{J}}} + E_{\mathbf{s}'}$ with $x_t = x(\mathbf{i} + 1_{\overline{\mathcal{J}}})$ with the property that there exists some t_s ($0 \leq t_s \leq t$) such that*

(a) $x_{t_s} = x(\mathbf{i}) + E_{\overline{\mathcal{J}}} + E_{\mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})}$, and

(b) $\chi_{k_r}(x_n) = \chi_{k_r}(x(\mathbf{i} + 1_{\overline{\mathcal{J}}}))$ for any $t_s \leq n \leq t$, or, $(x_n + l'_{[k]}, E_{j(n)}) = 1$ for $t_s \leq n < t$.

(II) *Let $\tilde{\mathbf{s}}$ be a subset of \mathcal{J}^* such that*

$$(3.2.7) \quad \chi_{k_r}(x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \tilde{\mathbf{s}}}) = \chi_{k_r}(x(\mathbf{i} + 1_{\overline{\mathcal{J}}})).$$

Then $\tilde{\mathbf{s}} \subseteq \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})$. Moreover, there exists a computation sequence $\{x_n\}_{n=0}^t$ as in Lemma 3.1.3 connecting $x_0 = x(\mathbf{i}) + E_{\overline{\mathcal{J}}}$ with $x_t = x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \tilde{\mathbf{s}}}$ such that $\chi_{k_r}(x_{n+1}) \leq \chi_{k_r}(x_n)$ for any $0 \leq n < t$.

(III) *For any cycle $l^* > 0$ with support $|l^*| \subseteq \mathcal{J}^* \setminus \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})$, there exists a computation sequence $\{x_n\}_{n=0}^t$ with $x_{n+1} = x_n + E_{j(n)}$ (for $n < t$), $x_0 = x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})}$ and $x_t = x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})} + l^*$ such that $\chi_{k_r}(x_{n+1}) \geq \chi_{k_r}(x_n)$ for any $0 \leq n < t$.*

Proof. (I) We will use the following notation: for any $x \geq x(\mathbf{i}) + E_{\overline{\mathcal{J}}}$ we write $\|x\|$ for the support $|x - x(\mathbf{i}) - E_{\overline{\mathcal{J}}}|$. Note that Lemma 3.1.3 guarantees the existence of a computation sequence connecting $x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \mathbf{s}'}$ with $x(\mathbf{i} + 1_{\overline{\mathcal{J}}})$. We consider such a sequence $\{x_n\}_{n=0}^t$ constructed in such a way that in the procedure of choices of $j(n)$'s we try to make $\|x_n\|$ as large as possible. More precisely, for any $0 \leq n < t_1$, the index $j(n) \in \mathcal{J}^*$ is chosen as follows:

$$(3.2.8) \quad \begin{cases} (x_n + l'_{[k]}, E_{j(n)}) > 0 \\ E_{j(n)} \notin \|x_n\|. \end{cases}$$

Assume that this stops for $n = t_1$, that is, for $n = t_1$ there is no index $j(n) \in \mathcal{J}^*$ which would satisfy (3.2.8). We claim that $\|x_{t_1}\| = \|x(\mathbf{i} + 1_{\overline{\mathcal{J}}})\| = \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})$, hence $t_s = t_1$ satisfies part (a) of the proposition.

Indeed, assume that this is not the case. Then we continue the construction of the sequence, let $t_2 + 1$ be the first index when $\|x\|$ increases, that is $\|x_n\| = \|x_{t_1}\|$ for $t_1 \leq n \leq t_2$ and $\|x_{t_2+1}\| = \|x_{t_1}\| \cup \{j^*\} \neq \|x_{t_1}\|$ for some $j^* \in \mathcal{J}^*$. Hence $j^* = j(t_2)$.

Since $(x_{t_2} + l'_{[k]}, E_{j^*}) > 0$ and $(x_{t_1} + l'_{[k]}, E_{j^*}) \leq 0$, we get $(x_{t_2} - x(\mathbf{i}), E_{j^*}) > -(x(\mathbf{i}) + l'_{[k]}, E_{j^*}) \geq (x_{t_1} - x(\mathbf{i}), E_{j^*})$. Since $x_{t_2} - x(\mathbf{i})$ and $x_{t_1} - x(\mathbf{i})$ have the same support, which does not contain j^* , this strict inequality can happen only if $(x_{t_1} - x(\mathbf{i}), E_{j^*}) > 0$. By the same argument, in fact, there exists a connected component C of the reduced cycle $x_{t_1} - x(\mathbf{i})$ such that

$$(3.2.9) \quad ((x_{t_2} - x(\mathbf{i}))|_C, E_{j^*}) > (C, E_{j^*}) > 0.$$

Next, we analyze the restriction of the sequence $z_n := x_n - x(\mathbf{i})$ to C for $t_1 \leq n \leq t_2$. First note that $(z_n, E_{j(n)}) = (x_n + l'_{[k]}, E_{j(n)}) - (x(\mathbf{i}) + l'_{[k]}, E_{j(n)}) > 0$. If $E_{j(n)}$ is supported by C then it does not intersect any other components of $x_{t_1} - x(\mathbf{i})$, hence $(z_n|_C, E_{j(n)}) > 0$ too. Let us consider that subsequence \tilde{z}_* of $z_n|_C$ which is obtained from $z_n|_C$ by eliminating those steps from the computation sequence of $\{x_n\}_{n=t_1}^{t_2}$ which correspond to elements $j(n)$ not supported by C . Then the sequence starts with C , ends with $(x_{t_2} - x(\mathbf{i}))|_C$, it is the beginning of a Laufer sequence connecting the connected C with the fundamental cycle of C , but at the step t_2 one has $(z_{t_2}|_C, E_{j(t_2)}) \geq 2$, cf. (3.2.9).

Note also that the sequence $z_n|_C$ is reduced along $\overline{\mathcal{J}}$, hence along the procedure we do not add any base element from $\overline{\mathcal{J}}$, hence if we decrease the self-intersections of these vertices we will not modify the Laufer data along the sequence. Hence we can assume that C is a rational graph. But this contradicts the existence of \tilde{z}_* , cf. 3.2.4.

Part (b) uses the same argument. We fix a connected component of $x_{t_s} - x(\mathbf{i})$. Since in the Laufer steps the components do not interact, we can even assume that the support of $x_{t_s} - x(\mathbf{i})$ is connected. Then $x_n - x(\mathbf{i})$ for $n \geq t_s$ is part of the computations sequence connecting the reduced connected $x_{t_s} - x(\mathbf{i})$ to its fundamental cycle. Since we may assume that C is rational (since the steps do not involve $\overline{\mathcal{J}}$), along the sequence we must have $(x_n - x(\mathbf{i}), E_{j(n)}) = 1$ by 3.2.4. This happens only if $(x_n + l'_{[k]}, E_{j(n)}) = 1$ and $(x(\mathbf{i}) + l'_{[k]}, E_{j(n)}) = 0$.

(II) Assume that $\tilde{\mathbf{s}} \not\subseteq \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})$, and set $\mathbf{s}' := \tilde{\mathbf{s}} \cap \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})$ and $\Delta \mathbf{s} := \tilde{\mathbf{s}} \setminus \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})$. Take a computation sequence $\{x_n\}_{n=0}^t$ as in (I) connecting $x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \mathbf{s}'}$ with $x(\mathbf{i} + 1_{\overline{\mathcal{J}}})$. Since $\chi_{k_r}(x_n)$ is non-increasing, $1 - (E_{j(n)}, x_n + l'_{[k]}) \leq 0$; see also 3.1.3. Therefore, $1 - (E_{j(n)}, x_n + E_{\Delta \mathbf{s}} + l'_{[k]}) \leq 0$ too, since $j(n) \notin \Delta \mathbf{s}$. Since $\{x_n + E_{\Delta \mathbf{s}}\}_n$ connects $x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \tilde{\mathbf{s}}}$ with $x(\mathbf{i} + 1_{\overline{\mathcal{J}}}) + E_{\Delta \mathbf{s}}$, we get

$$\chi_{k_r}(x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \tilde{\mathbf{s}}}) \geq \chi_{k_r}(x(\mathbf{i} + 1_{\overline{\mathcal{J}}}) + E_{\Delta \mathbf{s}}).$$

This together with assumption (3.2.7) and Lemma 3.2.3 guarantee that, in fact,

$$(3.2.10) \quad \chi_{k_r}(x(\mathbf{i} + 1_{\overline{\mathcal{J}}}) + E_{\Delta \mathbf{s}}) = \chi_{k_r}(x(\mathbf{i} + 1_{\overline{\mathcal{J}}})).$$

On the other hand,

$$\chi_{k_r}(x(\mathbf{i} + 1_{\overline{\mathcal{J}}}) + E_{\Delta \mathbf{s}}) - \chi_{k_r}(x(\mathbf{i} + 1_{\overline{\mathcal{J}}})) = \chi_{can}(E_{\Delta \mathbf{s}}) - (E_{\Delta \mathbf{s}}, x(\mathbf{i} + 1_{\overline{\mathcal{J}}}) + l'_{[k]}) \geq \chi_{can}(E_{\Delta \mathbf{s}}),$$

where the last inequality follows from the definition of $x(\mathbf{i} + 1_{\overline{\mathcal{J}}})$. Since $\chi_{can}(E_{\Delta \mathbf{s}})$ is the number of connected components of $E_{\Delta \mathbf{s}}$, it is strictly positive, a fact which contradicts (3.2.10).

The second part runs over similar argument. We construct a computation sequence as in (I), applied for $\mathbf{s}' = 0$, in such a way that first we choose only the $j(n)$'s from $\tilde{\mathbf{s}}$. We claim

that in this way we fill in all $\tilde{\mathbf{s}}$. Indeed, assume that this procedure stops at the level of x_m ; that is, $x(\mathbf{i}) + E_{\overline{\mathcal{J}}} \leq x_m < x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \tilde{\mathbf{s}}}$ and

$$(3.2.11) \quad (E_j, x_m + l'_{[k]}) \leq 0 \quad \text{for all } j \in \Delta \tilde{\mathbf{s}} := \tilde{\mathbf{s}} \setminus \|x_m\|.$$

Then

$$\chi_{k_r}(x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \tilde{\mathbf{s}}}) - \chi_{k_r}(x_m) = \chi_{can}(E_{\Delta \tilde{\mathbf{s}}}) - (E_{\Delta \tilde{\mathbf{s}}}, x_m + l'_{[k]}) \geq \chi_{can}(E_{\Delta \tilde{\mathbf{s}}}),$$

where the last inequality follows from (3.2.11). This, and the assumption (3.2.7) imply $\chi_{k_r}(x_m) < \chi_{k_r}(x(\mathbf{i} + 1_{\overline{\mathcal{J}}}))$, a fact which contradicts Lemma 3.2.3.

(III) The statement follows by induction from the following fact: if $l^* > 0$, $|l^*| \subseteq \mathcal{J}^* \setminus \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})$, then there exists $j \in |l^*|$ so that

$$\chi_{k_r}(x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})} + l^* - E_j) \leq \chi_{k_r}(x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})} + l^*).$$

Indeed, if not, then $(E_j, x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})} + l'_{[k]} + l^* - E_j) \geq 2$ for any $j \in |l^*|$. On the other hand, $(E_j, x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})} + l'_{[k]}) \leq 0$, by the proof of part (I) (namely, the choice of $t_s = t_1$), or by the definition of $\mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})$. Therefore, $(E_j, l^* - E_j) \geq 2$, or, $(E_j, l^* + k_{can}) \geq 0$ for all j . Summing up over the coefficients of l^* , we get $(l^*, l^* + k_{can}) \geq 0$, which contradicts (3.2.1) since the subgraph generated by $|l^*|$ is rational. \square

3.3. The lattice reduction. Now we are ready to formulate the main result of this section: in the definition of the lattice cohomology we wish to replace the (cubes of the) lattice L with cubes of a smaller rank free \mathbb{Z} -module associated with the bad vertices.

3.3.1. Definition of the (quadrant of the) new free \mathbb{Z} -module. Let us fix $[k]$ and assume that the graph G admits ν bad vertices as above. Then define $\overline{L} = (\mathbb{Z}_{\geq 0})^\nu$, and the function $\overline{w}_0 : (\mathbb{Z}_{\geq 0})^\nu \rightarrow \mathbb{Z}$ by

$$(3.3.2) \quad \overline{w}_0(i_1, \dots, i_\nu) := \chi_{k_r}(x(i_1, \dots, i_\nu)).$$

Then \overline{w}_0 defines a set $\{\overline{w}_q\}_{q=0}^\nu$ of compatible weight functions depending on $[k]$, defined similarly as in 2.3.3, denoted by $\overline{w}[k]$.

Theorem 3.3.3. (Reduction Theorem) *Let G be a negative definite connected graph and let k_r be the distinguished representative of a characteristic class. Suppose $\overline{\mathcal{J}} = \{j_k\}_{k=1}^\nu$ contains the bad vertices and $(\overline{L}, \overline{w}[k])$ is the first quadrant of the new weighted free \mathbb{Z} -module associated with $\overline{\mathcal{J}}$ and k_r . Then there is a graded $\mathbb{Z}[U]$ -module isomorphism*

$$(3.3.4) \quad \mathbb{H}^*(G, k_r) \cong \mathbb{H}^*(\overline{L}, \overline{w}[k]).$$

Note that via Lemma 2.6.2, (3.3.4) is equivalent to the isomorphism:

$$(3.3.5) \quad \mathbb{H}^*([0, \infty)^s, k_r) \cong \mathbb{H}^*([0, \infty)^\nu, \overline{w}[k]).$$

4. THE PROOF OF REDUCTION THEOREM

4.1. Notations, assumption. In this section we abbreviate k_r into k , $\overline{w}[k]$ into \overline{w} .

Assume that there exists a pair $j, j' \in \overline{\mathcal{J}}$, $j \neq j'$, such that $(E_j, E_{j'}) = 1$. Then we can blow up the intersection point $E_j \cap E_{j'}$. We have to observe two facts. First, the lattice cohomology $\mathbb{H}^*(G, k)$ is stable with respect to this blow up [16, 18]. Second, the ‘strict transform’ of the set $\overline{\mathcal{J}}$ can serve for a new set of bad vertices and the right hand side of (3.3.4) stays stable as well. Therefore, by additional blow ups, *we can assume that*

$$(E_j, E_{j'}) = 0 \quad \text{for every pair } j, j' \in \overline{\mathcal{J}}, j \neq j'.$$

4.2. The first step. Comparing S_N and \overline{S}_N . We consider the projections $\phi : (\mathbb{Z}_{\geq 0})^s \rightarrow (\mathbb{Z}_{\geq 0})^\nu$ and $\phi : [0, \infty)^s \rightarrow [0, \infty)^\nu$ given by $(m_j)_{j \in \mathcal{J}} \mapsto (m_j)_{j \in \overline{\mathcal{J}}}$. This induces a projection of the cubes too. If $(l, I) \in \mathcal{Q}(L)$ is a cube of L , then write I as $\overline{I} \cup I^*$ where $\overline{I} = I \cap \overline{\mathcal{J}}$ and $I^* = I \cap \mathcal{J}^*$. Then the vertices of (l, I) are projected via ϕ into the vertices of the cube $(\phi(l), \overline{I}) \in \mathcal{Q}(\overline{L})$ of \overline{L} . It is convenient to write $\overline{I} := \phi(I)$ and $\phi(l, I) := (\phi(l), \overline{I})$.

By 3.2.3, we get that for any $l \in (\mathbb{Z}_{\geq 0})^s$ we have $w(l) \geq \overline{w}(\phi(l))$, hence

$$(4.2.1) \quad w((l, I)) \geq \overline{w}(\phi(l, I)) \quad \text{for any cube } (l, I) \in \mathcal{Q}(L).$$

Recall that for any N we define $S_N \subseteq [0, \infty)^s$ as the union of cubes of $[0, \infty)^s$ of weight $\leq N$. Similarly, let $\overline{S}_N \subseteq [0, \infty)^\nu$ be the union of cubes $(\mathbf{i}, \overline{I})$ with $\overline{w}(\mathbf{i}, \overline{I}) \leq N$. Then, the statement of Theorem 3.3.3, via Theorem 2.4.1, is equivalent to the fact that

$$(4.2.2) \quad S_N \text{ and } \overline{S}_N \text{ have the same cohomology groups for any integer } N.$$

Note that by (4.2.1) $\phi(S_N) \subseteq \overline{S}_N$, and by construction $\phi|_{S_N} : S_N \rightarrow \overline{S}_N$ is a cubical map. For any $(\mathbf{i}, \overline{I}) \subseteq \overline{S}_N$ we consider $\phi_N^*(\mathbf{i}, \overline{I}) \subseteq S_N$ defined as the union of all cubes $(l, I) \subseteq S_N$ with $\phi(l, I) = (\mathbf{i}, \overline{I})$. [We warn the reader that this is not the inverse image $(\phi|_{S_N})^{-1}(\mathbf{i}, \overline{I})$, rather it is the closure of the inverse image of the interior of the cube $(\mathbf{i}, \overline{I})$; see also below.] If $\psi : [0, \infty)^s \rightarrow [0, \infty)^{s-\nu}$ is the second projection on the \mathcal{J}^* -coordinate direction, then $\phi_N^*(\mathbf{i}, \overline{I})$ is the product of $\psi(\phi_N^*(\mathbf{i}, \overline{I}))$ with the cube $(\mathbf{i}, \overline{I})$; in particular, it has the homotopy type of $\psi(\phi_N^*(\mathbf{i}, \overline{I}))$.

A Mayer–Vietoris inductive (or Leray type spectral sequence) argument shows that (4.2.2) follows from

$$(4.2.3) \quad \phi_N^*(\mathbf{i}, \overline{I}) \text{ is non-empty and contractible for any } (\mathbf{i}, \overline{I}) \in \overline{S}_N.$$

4.3. Generalities about contractions. In the sequel we fix a cube $(\mathbf{i}, \overline{I})$ from \overline{S}_N and we start to prove (4.2.3). For any such cube $(\mathbf{i}, \overline{I})$ we also consider the inverse image $\phi^{-1}(\mathbf{i}, \overline{I})$ consisting of the union of all cubes (l, I) of $[0, \infty)^s$ with $\phi(l, I) \subseteq (\mathbf{i}, \overline{I})$ (not necessarily from S_N). We can also consider $(\phi|_{S_N})^{-1}(\mathbf{i}, \overline{I})$, the union of cubes (l, I) from S_N with $\phi(l, I) \subseteq (\mathbf{i}, \overline{I})$. Clearly,

$$\phi_N^*(\mathbf{i}, \overline{I}) \subseteq (\phi|_{S_N})^{-1}(\mathbf{i}, \overline{I}) \subseteq \phi^{-1}(\mathbf{i}, \overline{I}).$$

Note that $\phi^{-1}(\mathbf{i}, \overline{I})$ is the product of the cube $(\mathbf{i}, \overline{I})$ with $[0, \infty)^{s-\nu}$. Our goal is to contract this ‘fiber direction space’ $[0, \infty)^{s-\nu}$ in such a way that along the contraction χ_k does not increase, and the contraction preserves the subspaces $\phi_N^*(\mathbf{i}, \overline{I})$ and $(\phi|_{S_N})^{-1}(\mathbf{i}, \overline{I})$ as well.

The cycles supported on \mathcal{J}^* (‘fiber direction’) will be denoted by $l^* = \sum_{j \in \mathcal{J}^*} m_j E_j$. For any pair l_1^* and l_2^* with $l_1^* \leq l_2^*$ we consider the real s -dimensional rectangle $R_{(\mathbf{i}, \overline{I})}(l_1^*, l_2^*)$, the product of a rectangle in the $(s - \nu)$ -dimensional space with the cube $(\mathbf{i}, \overline{I})$: it is the convex closure of the lattice points, which have the form

$$x(\mathbf{i}) + E_{\overline{\mathcal{J}}} + l^* \quad \text{with } \overline{\mathcal{J}} \subseteq \overline{I} \quad \text{and } l^* \in L, \quad l_1^* \leq l^* \leq l_2^*.$$

We extend this notation allowing l_2^* to have all its entries ∞ .

Note that the lattice points $x(\mathbf{i}) + E_{\overline{\mathcal{J}}} + l^*$, being in $[0, \infty)^s$, are effective, hence the relevant l^* satisfies $l^* \geq l_{1, \min}^* := -x(\mathbf{i}) + \sum_{j \in \overline{\mathcal{J}}} i_j E_j$ (the projection of $-x(\mathbf{i})$ on the \mathcal{J}^* -components). In particular, $R_{(\mathbf{i}, \overline{I})}(l_{1, \min}^*, \infty) = \phi^{-1}(\mathbf{i}, \overline{I}) \subseteq [0, \infty)^s$, and we can assume that l_1^* and l_2^* satisfy $l_{1, \min}^* \leq l_1^* \leq l_2^* \leq \infty$. Note also that $l_{1, \min}^* \leq 0$.

We start to discuss the existence of a contraction $c : R_{(\mathbf{i}, \overline{I})}(l_1^*, l_2^* + E_j) \rightarrow R_{(\mathbf{i}, \overline{I})}(l_1^*, l_2^*)$ for some $j \in \mathcal{J}^*$, acting in the direction of the \mathcal{J}^* -coordinates and having the property that χ_k will not increase along it. The map c is defined as follows. If a lattice point l is in $R_{(\mathbf{i}, \overline{I})}(l_1^*, l_2^*)$, then $c(l) = l$. Otherwise it has the form $l = x(\mathbf{i}) + E_{\overline{\mathcal{J}}} + l^* + E_j$ for some l^* with

$l_1^* \leq l^* \leq l_2^*$ and $m_j(l^*) = m_j(l_2^*)$. Then set $c(l) = l - E_j$. The next criterion guarantees that χ_k is not increasing along this contraction.

Lemma 4.3.1. *Assume that for some l_2^* and $j \in \mathcal{J}^*$ one has*

$$\chi_k(x(\mathbf{i}) + E_{\overline{T}} + l_2^* + E_j) \geq \chi_k(x(\mathbf{i}) + E_{\overline{T}} + l_2^*).$$

Then, for any l^ with $l_1^* \leq l^* \leq l_2^*$ and $m_j(l^*) = m_j(l_2^*)$, and for every $\overline{J} \subseteq \overline{T}$, one also has*

$$\chi_k(x(\mathbf{i}) + E_{\overline{J}} + l^* + E_j) \geq \chi_k(x(\mathbf{i}) + E_{\overline{J}} + l^*).$$

Therefore, $\chi_k(c(l)) \leq \chi_k(l)$ for any $l \in R_{(\mathbf{i}, \overline{T})}(l_1^, l_2^* + E_j)$.*

Proof. Use $\chi_k(z + E_j) = \chi_k(z) + 1 - (E_j, z + l'_{[k]})$ and $(E_j, E_{\overline{T}} - E_{\overline{J}} + l_2^* - l^*) \geq 0$. \square

The following lemma generalizes results of [16, § 3.2], where the case $\nu = 1$ is treated.

Lemma 4.3.2. *Assume that for some fixed l_2^* there exists an infinite sequence of cycles $\{x_n^*\}_{n \geq 0}$, $x_n^* = \sum_{j \in \mathcal{J}^*} m_{j,n} E_j$, with $x_0^* = l_2^*$ such that*

- (a) $x_{n+1}^* = x_n^* + E_{j(n)}$ for some $j(n) \in \mathcal{J}^*$, $n \geq 0$;
- (b) $\chi_k(x(\mathbf{i}) + E_{\overline{T}} + x_{n+1}^*) \geq \chi_k(x(\mathbf{i}) + E_{\overline{T}} + x_n^*)$ for any $n \geq 0$.
- (c) for any fixed j the sequence $m_{j,n}$ tends to infinity as n tends to infinity;

Then there exists a contraction of $R_{(\mathbf{i}, \overline{T})}(l_1^, \infty)$ to $R_{(\mathbf{i}, \overline{T})}(l_1^*, l_2^*)$ along which χ_k is non-increasing.*

Proof. Use Lemma 4.3.1 and induction over n . \square

Symmetrically, by similar proof, one has the following statements too.

Lemma 4.3.3.

(I) *For any fixed l_1^* and $j \in \mathcal{J}^*$ with $l_1^* - E_j \geq l_{1,min}^*$ if*

$$\chi_k(x(\mathbf{i}) + l_1^* - E_j) \geq \chi_k(x(\mathbf{i}) + l_1^*),$$

then for any l^ with $l_1^* \leq l^* \leq l_2^*$ and $m_j(l^*) = m_j(l_1^*)$, and for every $\overline{J} \subseteq \overline{T}$, one also has*

$$\chi_k(x(\mathbf{i}) + E_{\overline{J}} + l^* - E_j) \geq \chi_k(x(\mathbf{i}) + E_{\overline{J}} + l^*).$$

Therefore, $R_{(\mathbf{i}, \overline{T})}(l_1^ - E_j, l_2^*)$ contracts onto $R_{(\mathbf{i}, \overline{T})}(l_1^*, l_2^*)$ such that χ_k is non-increasing along the contraction.*

(II) *Assume that there exists a sequence of cycles $\{x_n^*\}_{n=0}^t$ with $x_0^* = l_{1,min}^*$ and $x_t^* = l_1^*$ such that for any $0 \leq n < t$ one has*

- (a) $x_{n+1}^* = x_n^* + E_{j(n)}$ for some $j(n) \in \mathcal{J}^*$,
- (b) $\chi_k(x(\mathbf{i}) + x_n^*) \geq \chi_k(x(\mathbf{i}) + x_{n+1}^*)$.

Then there exists a contraction of $R_{(\mathbf{i}, \overline{T})}(l_{1,min}^, l_2^*)$ to $R_{(\mathbf{i}, \overline{T})}(l_1^*, l_2^*)$ along which χ_k is non-increasing.*

4.4. Contractions. In this subsection we apply the results of the previous subsection 4.3 in order to contract the triple $(\phi^{-1}(\mathbf{i}, \overline{T}), (\phi|_{S_N})^{-1}(\mathbf{i}, \overline{T}), \phi_N^*(\mathbf{i}, \overline{T}))$.

First we show the existence of a sequence of cycles $\{x_n^*\}_{n=0}^t$ with $x_0^* = l_{1,min}^*$ and $x_t^* = 0$ which satisfies the assumptions of Lemma 4.3.3(II). This follows inductively from the following lemma.

Lemma 4.4.1. *For any x^* with $l_{1,min}^* \leq x^* < 0$ and supported on \mathcal{J}^* there exists at least one index $j \in |x^*|$ such that*

$$(4.4.2) \quad \chi_k(x(\mathbf{i}) + x^*) \geq \chi_k(x(\mathbf{i}) + x^* + E_j).$$

Proof. (4.4.2) is equivalent to $(E_j, x(\mathbf{i}) + l'_{[k]} + x^*) \geq 1$ for some $j \in |x^*|$. Otherwise $(E_j, x(\mathbf{i}) + l'_{[k]} + x^*) \leq 0$ for every $j \in |x^*|$. On the other hand, for $j \in \mathcal{J}^* \setminus |x^*|$ one has $(E_j, x^*) \leq 0$ and $(E_j, x(\mathbf{i}) + l'_{[k]}) \leq 0$ by 3.1.2(b). Hence $(E_j, x(\mathbf{i}) + l'_{[k]} + x^*) \leq 0$ for every $j \in \mathcal{J}^*$. This contradicts the minimality of $x(\mathbf{i})$ in 3.1.2(c). \square

In particular, Lemma 4.3.3(II) applies for $l_1^* = 0$ and any $l_2^* \geq 0$ (including ∞).

Next, we search for a convenient small cycle l_2^* for which Lemma 4.3.2 applies as well. First we show that $l_2^* = \infty$ can be replaced by $x(\mathbf{i} + 1_{\overline{\mathcal{I}}}) - x(\mathbf{i}) - E_{\overline{\mathcal{I}}}$.

Lemma 4.4.3. *There exists a sequence as in Lemma 4.3.2 with $x_0^* = x(\mathbf{i} + 1_{\overline{\mathcal{I}}}) - x(\mathbf{i}) - E_{\overline{\mathcal{I}}}$.*

Proof. First we show the existence of some l_2^* , with all its coefficient very large, which can be connected by a computation sequence to ∞ with properties (a)-(b)-(c) of 4.3.2. For this, consider the full subgraph supported by \mathcal{J}^* . Since it is negative definite, it supports an effective cycle Z^* such that $(Z^*, E_j) < 0$ for any $j \in \mathcal{J}^*$. Consider any sequence $\{x_n^*\}_{n=0}^t$, $x_{n+1}^* = x_n^* + E_{j(n)}$, such that $x_0^* = 0$ and $x_t^* = Z^*$. Then, there exists $\ell_0 \geq 1$ sufficiently large such that for any $\ell \geq \ell_0$ and n one has

$$\chi_k(x(\mathbf{i}) + E_{\overline{\mathcal{I}}} + \ell Z^* + x_{n+1}^*) \geq \chi_k(x(\mathbf{i}) + E_{\overline{\mathcal{I}}} + \ell Z^* + x_n^*).$$

Hence the sequence $\{\ell Z + x_n\}_{\ell \geq \ell_0, 0 \leq n \leq t}$ connects $l_2^* = \ell_0 Z^*$ with ∞ with the required properties.

Next, we connect $x(\mathbf{i} + 1_{\overline{\mathcal{I}}}) - x(\mathbf{i}) - E_{\overline{\mathcal{I}}}$ with this l_2^* via a sequence which satisfies (a)-(b)-(c). Its existence follows from the following statement:

For any $l^* > 0$ supported by \mathcal{J}^* there exists at least one index $j \in |l^*|$ such that

$$\chi_k(x(\mathbf{i} + 1_{\overline{\mathcal{I}}}) + l^* - E_j) \leq \chi_k(x(\mathbf{i} + 1_{\overline{\mathcal{I}}}) + l^*).$$

Indeed, assume the opposite. Then $(E_j, l^*) \geq E_j^2 + 2$ for any $j \in |l^*|$. Hence $(E_j, l^* + k_{can}) \geq 0$, or $\chi(l^*) \leq 0$, which contradict the rationality of the subgraph supported by \mathcal{J}^* . \square

Note that by Proposition 3.2.6(I), the ‘upper’ bound $l_2^* = x(\mathbf{i} + 1_{\overline{\mathcal{I}}}) - x(\mathbf{i}) - E_{\overline{\mathcal{I}}}$ can be pushed down further to its support $\mathbf{s}(\mathbf{i}, \overline{\mathcal{I}})$. Hence 3.2.6(I), 4.4.3 and 4.4.1 imply the following.

Corollary 4.4.4. *There exists a deformation contraction of $\phi^{-1}(\mathbf{i}, \overline{\mathcal{I}})$ to $R_{(\mathbf{i}, \overline{\mathcal{I}})}(0, E_{\mathbf{s}(\mathbf{i}, \overline{\mathcal{I}})})$ along which χ_k is non-increasing. Moreover, its restriction induces a deformation retract from $(\phi|_{S_N})^{-1}(\mathbf{i}, \overline{\mathcal{I}})$ to $S_N \cap R_{(\mathbf{i}, \overline{\mathcal{I}})}(0, E_{\mathbf{s}(\mathbf{i}, \overline{\mathcal{I}})})$. Restricting further, it gives a deformation retract from $\phi_N^*(\mathbf{i}, \overline{\mathcal{I}})$ to $\Phi_N^*(\mathbf{i}, \overline{\mathcal{I}})$, where $\Phi_N^*(\mathbf{i}, \overline{\mathcal{I}})$ is the product of the cube $(\mathbf{i}, \overline{\mathcal{I}})$ with*

$$\psi(\phi_N^*(\mathbf{i}, \overline{\mathcal{I}})) \cap \{l^* : 0 \leq l^* - \psi(x(\mathbf{i})) \leq E_{\mathbf{s}(\mathbf{i}, \overline{\mathcal{I}})}\}.$$

Note that this last space $\Phi_N^*(\mathbf{i}, \overline{\mathcal{I}})$ is now rather ‘small’: it is contained in the cube $(x(\mathbf{i}), \overline{\mathcal{I}} \cup \mathbf{s}(\mathbf{i}, \overline{\mathcal{I}}))$. Nevertheless, the N -filtration of this cube can be rather complicated.

The statement of the above corollary means that if $\Phi_N^*(\mathbf{i}, \overline{\mathcal{I}})$ is empty then $\phi_N^*(\mathbf{i}, \overline{\mathcal{I}})$ is empty too, otherwise they have the same homotopy type. Therefore, via (4.2.3), we need to show that

$$\Phi_N^*(\mathbf{i}, \overline{\mathcal{I}}) \text{ is non-empty and contractible.}$$

4.5. The non-emptiness of $\Phi_N^*(\mathbf{i}, \overline{\mathcal{I}})$. Recall that we fixed an integer N and a cube $(\mathbf{i}, \overline{\mathcal{I}})$ which belongs to \overline{S}_N . By Definition 3.3.1 and Proposition 3.2.6 this reads as

$$(4.5.1) \quad \chi_k(x(\mathbf{i} + 1_{\overline{\mathcal{J}}})) = \chi_k(x(\mathbf{i}) + E_{\overline{\mathcal{J}} \cup \mathbf{s}(\mathbf{i}, \overline{\mathcal{J}})}) \leq N \text{ for every } \overline{\mathcal{J}} \subseteq \overline{\mathcal{I}}.$$

Proposition 4.5.2. *For any fixed cube $(\mathbf{i}, \bar{I}) \in \bar{S}_N$ there exists a cycle in L of the form $x(\mathbf{i}) + E_{\tilde{\mathbf{s}}(\mathbf{i}, \bar{I})}$ such that $\tilde{\mathbf{s}}(\mathbf{i}, \bar{I}) \subseteq \mathbf{s}(\mathbf{i}, \bar{I})$ and $(x(\mathbf{i}) + E_{\tilde{\mathbf{s}}(\mathbf{i}, \bar{I})}, \bar{I}) \subseteq \Phi_N^*(\mathbf{i}, \bar{I})$; that is*

$$(4.5.3) \quad \chi_k(x(\mathbf{i}) + E_{\bar{\mathcal{J}} \cup \tilde{\mathbf{s}}(\mathbf{i}, \bar{I})}) \leq N \quad \text{for every } \bar{\mathcal{J}} \subseteq \bar{I}.$$

Proof. The proof is rather long, it fills all this subsection. It is an induction over the cardinality of $\bar{\mathcal{J}}$, respectively of \bar{I} . Before we start it, we perform a *few reductions* to simplify the involved combinatorial complexity. We will also write $\tilde{\mathbf{s}} := \tilde{\mathbf{s}}(\mathbf{i}, \bar{I})$.

4.5.4. First Reduction: $\bar{I} = \bar{\mathcal{J}}$. Consider $\bar{\mathcal{J}} \setminus \bar{I} = \bar{I}^c$ and the graph $G \setminus \bar{I}^c$ obtained from the original graph G by deleting the vertices \bar{I}^c and adjacent edges. The connected components of $G \setminus \bar{I}^c$ do not interact from the point of view of the statement of the above theorem. Indeed, the Laufer algorithm does not propagate along the bad vertices \bar{I}^c , and it is also enough to find supports $\tilde{\mathbf{s}}$ for each component independently. Hence, *we may assume that $\bar{I} = \bar{\mathcal{J}}$.*

4.5.5. Reformulation. Define (cf. Proposition 3.2.6(I))

$$(4.5.6) \quad N(G) := \max_{\bar{\mathcal{J}} \subseteq \bar{I}} \chi_k(x(\mathbf{i} + 1_{\bar{\mathcal{J}}})) = \max_{\bar{\mathcal{J}} \subseteq \bar{I}} \chi_k(x(\mathbf{i}) + E_{\bar{\mathcal{J}} \cup \mathbf{s}(\mathbf{i}, \bar{\mathcal{J}})}).$$

$N(G)$ is the smallest integer N for which (4.5.1) is valid; hence it is enough to prove Theorem 4.5.2 only for $N = N(G)$. Note that $N(G)$ depends on (\mathbf{i}, \bar{I}) , though in its notation this is not emphasized.

In fact, even the weight $\chi_k(x(\mathbf{i}))$ — and partly the cycle $x(\mathbf{i})$, cf. 4.5.9, — are irrelevant in the sense that it is enough to treat a relative version of the statement. Indeed, we can consider only the value $\Delta N(G) := N(G) - \chi_k(x(\mathbf{i}))$, which equals (use the last term of (4.5.6)):

$$(4.5.7) \quad \Delta N(G) = \max_{\bar{\mathcal{J}} \subseteq \bar{I}} \left(\chi_{\text{can}}(E_{\bar{\mathcal{J}} \cup \mathbf{s}(\mathbf{i}, \bar{\mathcal{J}})}) - (E_{\bar{\mathcal{J}} \cup \mathbf{s}(\mathbf{i}, \bar{\mathcal{J}})}, x(\mathbf{i}) + l'_{[k]}) \right).$$

Then we have to find $\tilde{\mathbf{s}} \subseteq \mathbf{s}(\mathbf{i}, \bar{I})$, such that for any $\bar{\mathcal{J}} \subseteq \bar{I}$ one has

$$(4.5.8) \quad \chi_{\text{can}}(E_{\bar{\mathcal{J}} \cup \tilde{\mathbf{s}}}) - (E_{\bar{\mathcal{J}} \cup \tilde{\mathbf{s}}}, x(\mathbf{i}) + l'_{[k]}) \leq \Delta N(G).$$

Note also that for a reduced cycle Z of G (as $E_{\bar{\mathcal{J}} \cup \mathbf{s}(\mathbf{i}, \bar{\mathcal{J}})}$ or $E_{\bar{\mathcal{J}} \cup \tilde{\mathbf{s}}}$), $\chi_{\text{can}}(Z)$ is the number of components of Z , which sometimes will also be denoted by $\#(Z)$.

4.5.9. Final Reformulation. For any vertex j and $J \subseteq \bar{\mathcal{J}}$ set

$$\sigma_j := 1 - (E_j, x(\mathbf{i}) + l'_{[k]}) \quad \text{and} \quad \sigma_j(J) := \sigma_j - (E_j, E_J).$$

By definition of $x(\mathbf{i})$, one has $\sigma_j > 0$ for any $j \in \bar{\mathcal{J}}^*$. Note also that the information needed in (4.5.7) and (4.5.8) about $x(\mathbf{i}) + l'_{[k]}$ can be totally codified by the integers σ_j . This permits to reformulate the statement of the paragraph 4.5.5 into the following version:

Let G be a plumbing graph with $\bar{\mathcal{J}} = \bar{\mathcal{J}} \sqcup \bar{\mathcal{J}}^*$, such that any two vertices of $\bar{\mathcal{J}}$ are not adjacent, and with additional decorations $\{\sigma_j\}_{j \in \bar{\mathcal{J}}}$ where $\sigma_j > 0$ for $j \in \bar{\mathcal{J}}^*$. Fix $\bar{I} \subseteq \bar{\mathcal{J}}$. For each $\bar{\mathcal{J}} \subseteq \bar{I}$ we define $\mathbf{s}(\bar{\mathcal{J}})$ as the minimal support in $\bar{\mathcal{J}}^*$ such that for any $j \in \bar{\mathcal{J}}^* \setminus \mathbf{s}(\bar{\mathcal{J}})$ one has $\sigma_j(\bar{\mathcal{J}} \cup \mathbf{s}(\bar{\mathcal{J}})) > 0$. [Clearly, $\mathbf{s}(\bar{\mathcal{J}})$ corresponds to $\mathbf{s}(\mathbf{i}, \bar{\mathcal{J}})$ in the original version, see also 3.2.6.]

Here $\mathbf{s}(\emptyset) = \emptyset$. The ‘modified’ Laufer algorithm to find $\mathbf{s}(\bar{\mathcal{J}})$ (transcribed in the language of σ_j ’s) is the following. We construct the sequence of supports $\{s_n\}_{n=0}^t$ by the next principle: $s_0 = \emptyset$, and if s_n is already constructed and there exists some $j(n) \in \bar{\mathcal{J}}^* \setminus s_n$ such that

$$(4.5.10) \quad \sigma_{j(n)}(\bar{\mathcal{J}} \cup s_n) = \sigma_{j(n)} - (E_{j(n)}, E_{\bar{\mathcal{J}} \cup s_n}) \leq 0$$

then take $s_{n+1} := s_n \cup j(n)$; otherwise stop, and set $t = n$. [This again follows from the fact that $(E_j, x(\mathbf{i}) + E_{\bar{\mathcal{J}} \cup s_n} + l'_{[k]}) > 0$ if and only if $\sigma_j(\bar{\mathcal{J}} \cup s_n) \leq 0$.] Then the statements form 4.5.5 read as follows.

For any $\bar{\mathcal{J}} \subseteq \bar{\mathcal{I}}$ set

$$(4.5.11) \quad \Delta(\bar{\mathcal{J}}; G) := \#(E_{\bar{\mathcal{J}} \cup \mathbf{s}(\bar{\mathcal{J}})}) + \sum_{j \in \bar{\mathcal{J}} \cup \mathbf{s}(\bar{\mathcal{J}})} (\sigma_j - 1), \quad \text{and} \quad \Delta N(G) = \max_{\bar{\mathcal{J}} \subseteq \bar{\mathcal{I}}} \Delta(\bar{\mathcal{J}}; G).$$

Then there exists $\tilde{\mathbf{s}} \subseteq \mathbf{s}(\bar{\mathcal{I}})$ which for any $\bar{\mathcal{J}} \subseteq \bar{\mathcal{I}}$ satisfies

$$(4.5.12) \quad \#(E_{\bar{\mathcal{J}} \cup \tilde{\mathbf{s}}}) + \sum_{j \in \bar{\mathcal{J}} \cup \tilde{\mathbf{s}}} (\sigma_j - 1) \leq \Delta N(G).$$

Note, cf. 4.5.4, that we also may assume that $\bar{\mathcal{I}} = \bar{\mathcal{J}}$.

4.5.13. Let us analyze how the numerical invariants are modified along the computation sequence $\{s_n\}_{n=0}^t$ of 4.5.9. Note that if (4.5.10) occurs, since $\sigma_{j(n)} > 0$, $j(n)$ should be adjacent to $\bar{\mathcal{J}} \cup s_n$. If it is adjacent to only one vertex of $\bar{\mathcal{J}} \cup s_n$, then necessarily $\sigma_{j(n)} = 1$. Furthermore, in any situation, $\#(E_{\bar{\mathcal{J}} \cup s_n})$ is decreasing by $(E_{j(n)}, E_{\bar{\mathcal{J}} \cup s_n}) - 1$. Therefore, the sequence $a_n(\bar{\mathcal{J}}) := \#(E_{\bar{\mathcal{J}} \cup s_n}) + \sum_{j \in \bar{\mathcal{J}} \cup s_n} (\sigma_j - 1)$ is modified during this step by

$$a_{n+1}(\bar{\mathcal{J}}) - a_n(\bar{\mathcal{J}}) = \sigma_{j(n)} - (E_{j(n)}, E_{\bar{\mathcal{J}} \cup s_n}) \leq 0.$$

4.5.14. Second Reduction: $\sigma_j > 0$. Consider the reformulated situation from 4.5.9. Assume that $\sigma_j \leq 0$ for some $j \in \bar{\mathcal{I}} = \bar{\mathcal{J}}$, and consider the graph $G \setminus j$ obtained from G by deleting the vertex j and its adjacent edges. Note the following facts:

- The maximum $\Delta N(G)$ in (4.5.11) can be realized by a subset $\bar{\mathcal{J}}$ which does not contain j . In fact, for any $\bar{\mathcal{J}}$ with $j \notin \bar{\mathcal{J}}$ one has $\Delta(\bar{\mathcal{J}} \cup j; G) \leq \Delta(\bar{\mathcal{J}}; G)$. Indeed, using the notations from 4.5.13, $a_0(\bar{\mathcal{J}} \cup j) \leq a_0(\bar{\mathcal{J}})$; the sequence s_n associated with $\bar{\mathcal{J}}$ is good as the beginning of the sequence of $\bar{\mathcal{J}} \cup j$, and during this inductive steps $a_n(\bar{\mathcal{J}} \cup j)$ drops more than $a_n(\bar{\mathcal{J}})$; and finally, if the sequence of $\bar{\mathcal{J}} \cup j$ is longer, then its a_n -values decrease even more (cf. 4.5.13).

- All the support $\mathbf{s}(\bar{\mathcal{J}})$ definitely are included in $G \setminus j$ (since are subsets of \mathcal{J}^*).
- If we find for each component of $G \setminus j$ some $\tilde{\mathbf{s}}$ satisfying the statements of the theorem for that component, then their union solves the problem for G as well.

Therefore, having G with some $\sigma_j \leq 0$, we can delete j and continue to search for $\tilde{\mathbf{s}}$ for $G \setminus j$: that support will work for G as well.

If we delete all vertices with $\sigma_j \leq 0$ ($j \in \bar{\mathcal{I}}$) then we arrive to a situation when $\sigma_j > 0$ for any $j \in \bar{\mathcal{I}}$, hence, a posteriori, $\sigma_j > 0$ for any $j \in \mathcal{J}$.

Note that the wished reformulated statement from 4.5.9, even for all $\sigma_j = 1$, when the problem depends purely on the shape of the graph, is far to be trivial.

4.5.15. In this paragraph we list two statements regarding the integers $\Delta(\bar{\mathcal{J}}, G)$ and $\Delta N(G)$ which will be used in the body of the proof.

Claim A. For any $\bar{\mathcal{J}} \subseteq \bar{\mathcal{I}}$ and vertex $j \in \bar{\mathcal{I}} \setminus \bar{\mathcal{J}}$ one has

$$\Delta(\bar{\mathcal{J}} \cup j; G) = \Delta(\bar{\mathcal{J}}; G) + \sigma_j - (E_j, E_{\mathbf{s}(\bar{\mathcal{J}})}).$$

Its proof runs as follows. Let $\{s_n\}_{n=0}^t$ be the computation sequence for $\mathbf{s}(\bar{\mathcal{J}})$. It can be considered as the first part of a sequence for $\mathbf{s}(\bar{\mathcal{J}} \cup j)$ too; let $\{s_n\}_{n=t+1}^{t'}$ be its continuation for $\mathbf{s}(\bar{\mathcal{J}} \cup j)$. The coefficients $a_n(\bar{\mathcal{J}})$ and $a_n(\bar{\mathcal{J}} \cup j)$ for $n \leq t$ can be compared. Indeed, $a_0(\bar{\mathcal{J}} \cup j) = a_0(\bar{\mathcal{J}}) + \sigma_j$, and by 4.5.13, $a_t(\bar{\mathcal{J}} \cup j) = a_t(\bar{\mathcal{J}}) + \sigma_j - (E_j, E_{\mathbf{s}(\bar{\mathcal{J}})})$, the right hand side of the above identity (since $a_t(\bar{\mathcal{J}}) = \Delta(\bar{\mathcal{J}}; G)$).

Next, we show that $a_n(\bar{\mathcal{J}} \cup j)$ is constant for any further value $n \geq t$. For this define $s_n^j := s_n \setminus \mathbf{s}(\bar{\mathcal{J}})$, e.g. $s_t^j = \emptyset$. First let us analyze the step $n = t$. Then $\sigma_{j(t)} - (E_{j(t)}, E_{\bar{\mathcal{J}} \cup \mathbf{s}(\bar{\mathcal{J}})}) > 0$ (since $\mathbf{s}(\bar{\mathcal{J}})$ is completed), but $\sigma_{j(t)} - (E_{j(t)}, E_{\bar{\mathcal{J}} \cup \mathbf{s}(\bar{\mathcal{J}}) \cup j}) \leq 0$ (since $\mathbf{s}(\bar{\mathcal{J}} \cup j)$ is not completed). Hence $(E_j, E_{j(t)}) = 1$, and $a_{t+1}(\bar{\mathcal{J}} \cup j) = a_t(\bar{\mathcal{J}} \cup j)$.

In general, at every step, by induction, $E_{j \cup s_n^j}$ is connected, hence $(E_{j(n)}, E_{j \cup s_n^j})$ can be at most one (since the graph contains no loops). Hence, $\sigma_{j(n)} - (E_{j(n)}, E_{\bar{\mathcal{J}} \cup \mathbf{s}(\bar{\mathcal{J}})}) > 0$, and $\sigma_{j(n)} - (E_{j(n)}, E_{\bar{\mathcal{J}} \cup \mathbf{s}(\bar{\mathcal{J}}) \cup s_n^j \cup j}) \leq 0$ imply $(E_{j(n)}, E_{j \cup s_n^j}) = 1$ and $a_{n+1}(\bar{\mathcal{J}} \cup j) = a_n(\bar{\mathcal{J}} \cup j)$.

Claim B. Fix a vertex $\bar{j} \in \bar{\mathcal{I}}$ with $\sigma_{\bar{j}} \geq 1$, and assume that for all realizations of $\Delta N(G)$ as $\Delta(\bar{\mathcal{J}}, G)$ (as in (4.5.11)) one has $\bar{\mathcal{J}} \ni \bar{j}$. Let G_{-1} be the graph obtained from G by replacing the decoration $\sigma_{\bar{j}}$ by $\sigma_{\bar{j}} - 1$. We claim that

$$(4.5.16) \quad \Delta N(G_{-1}) = \Delta N(G) - 1.$$

Indeed, since $\{\sigma_j\}_{j \in \mathcal{J}^*}$ is unmodified, the support $\mathbf{s}(\bar{\mathcal{J}})$ for any $\bar{\mathcal{J}}$ is the same determined in G_{-1} or in G . If $\bar{\mathcal{J}} \not\ni \bar{j}$ then $\Delta(\bar{\mathcal{J}}, G_{-1}) = \Delta(\bar{\mathcal{J}}, G)$ by (4.5.11), hence $\Delta(\bar{\mathcal{J}}, G_{-1}) < \Delta N(G)$. If $\bar{\mathcal{J}} \ni \bar{j}$ then $\Delta(\bar{\mathcal{J}}, G_{-1}) = \Delta(\bar{\mathcal{J}}, G) - 1$ by the same (4.5.11). Since one such $\bar{\mathcal{J}}$ realizes $\Delta N(G)$, the claim follows.

4.5.17. Third Reduction: $G = G^-$. Assume $\bar{\mathcal{I}} = \bar{\mathcal{J}}$. Let G^- be the minimal connected subgraph of G generated by the vertices $\bar{\mathcal{I}}$. Here the vertices $\mathcal{J}(G^-)$ have an induced disjoint decomposition into $\bar{\mathcal{J}}(G^-) = \bar{\mathcal{J}}$ and $\mathcal{J}^*(G^-) = \mathcal{J}(G^-) \cap \mathcal{J}^*$. Moreover, each connected component of $G \setminus G^-$ is glued to G^- via a unique $j \in \bar{\mathcal{I}}$.

We claim that a solution $\tilde{\mathbf{s}}$ for G^- provides a solution for G too. Indeed, for any $\bar{\mathcal{J}} \subseteq \bar{\mathcal{I}}$, the supports $\mathbf{s}_G(\bar{\mathcal{J}})$ and $\mathbf{s}_{G^-}(\bar{\mathcal{J}})$ generated in G , respectively in G^- satisfy:

$\bar{\mathcal{J}} \cup \mathbf{s}_G(\bar{\mathcal{J}})$ can be obtained from $\bar{\mathcal{J}} \cup \mathbf{s}_{G^-}(\bar{\mathcal{J}})$ by gluing some subtrees of $G \setminus G^-$ along some elements of $\bar{\mathcal{J}}$. These subtrees are maximal among those connected subgraphs of G with all $\sigma_j = 1$ and adjacent to G^- . In particular, $\bar{\mathcal{J}} \cup \mathbf{s}_{G^-}(\bar{\mathcal{J}}) \subseteq \bar{\mathcal{J}} \cup \mathbf{s}_G(\bar{\mathcal{J}})$, and their topological realizations are homotopy equivalent; $\sigma_j = 1$ for any $j \in \mathbf{s}_G(\bar{\mathcal{J}}) \setminus \mathbf{s}_{G^-}(\bar{\mathcal{J}})$; and the integers $\#E_{\bar{\mathcal{J}} \cup \mathbf{s}(\bar{\mathcal{J}})}$ computed for G and G^- are the same.

Therefore, $\Delta N(G) = \Delta N(G^-)$, and a solution $\tilde{\mathbf{s}}$ for G^- is a solution for G too.

Hence, we can assume that $G = G^-$.

This ends the list of possible reductions/preparations and we start the ‘real’ proof.

4.5.18. The induction. The proof is based on inductive argument over such decorated graphs (with $G = G^-$, $\bar{\mathcal{I}} = \bar{\mathcal{J}}$, $\sigma_j > 0$), where we will consider subgraphs (with induced decorations σ_j), and eventually we will decrease the decorations $\{\sigma_j\}_{j \in \bar{\mathcal{J}}}$.

If $\bar{\mathcal{I}}$ is empty then $\Delta N(G) = 0$; if $\bar{\mathcal{I}}$ contains exactly one element j_0 , then by (4.5.17) $G = \{j_0\}$ and by (4.5.11) $\Delta N(G) = \sigma_{j_0}$. In both cases $\tilde{\mathbf{s}} = \emptyset$ answers the problem.

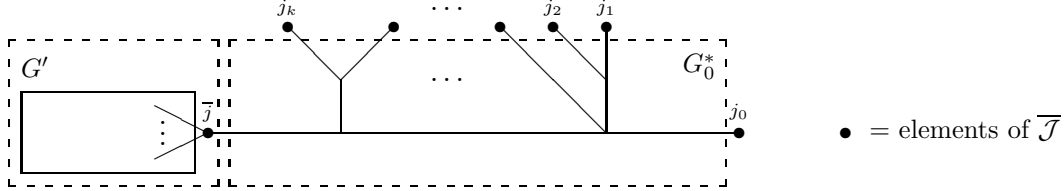
4.5.19. The inductive step is based on the following picture. Recall that G agrees with the smallest connected subgraph generated by $\bar{\mathcal{J}}$. Let $j_0 \in \bar{\mathcal{J}}$ be one of its end-vertices (that is, a vertex which has only one adjacent vertex in G). Denote that connected component of $G \setminus \bar{\mathcal{J}}$ which is adjacent to j_0 by G_0^* .

If $G \setminus \bar{\mathcal{J}} = G_0^*$ then all the vertices from $\bar{\mathcal{J}}$ are adjacent to G_0^* and $\bar{\mathcal{J}} = \bar{\mathcal{I}}$ is *exactly* the set of end-vertices of G . Then one verifies (use 4.5.15(A)) that

- $\Delta(\bar{\mathcal{J}}; G)$ is increasing function in $\bar{\mathcal{J}}$, hence $\Delta N(G) = \Delta(\bar{\mathcal{I}}, G)$, and
- $\#(E_{\bar{\mathcal{J}} \cup \mathbf{s}(\bar{\mathcal{J}})}) = \#(E_{\bar{\mathcal{I}} \cup \mathbf{s}(\bar{\mathcal{I}})})$, hence (4.5.12) holds for $\tilde{\mathbf{s}} = \mathbf{s}(\bar{\mathcal{I}})$.

Next, assume that $G \setminus \overline{\mathcal{J}} \neq G_0^*$. We may also assume (by a good choice of j_0) that there is *only one* vertex \bar{j} of $\overline{\mathcal{J}}$ which is simultaneously adjacent to G_0^* and to some other component of $G \setminus \overline{\mathcal{J}}$. Let $\{j_0, j_1, \dots, j_k, \bar{j}\}$ be the elements of $\overline{\mathcal{J}}$ which are adjacent to G_0^* . Then j_0, j_1, \dots, j_k are end-vertices of G . Let G' be obtained from G by deleting G_0^* , $\{j_0, j_1, \dots, j_k\}$ and all their adjacent edges.

Here is the schematic picture of G , where the vertices from \mathcal{J}^* are not emphasized:



The inductive step splits in several cases (**A** and **B**, **A** splits into **I** and **II**, while **I** has two subcases **I.a** and **I.b**).

4.5.20. A. Assume that $\Delta N(G)$ in (4.5.11) can be realized by some $\overline{\mathcal{J}}$ with $j_0 \notin \overline{\mathcal{J}}$.

Fix such a $\overline{\mathcal{J}}$. Since $\sigma_{j_0} \geq 1$ and $\Delta(\overline{\mathcal{J}}, G) \geq \Delta(\overline{\mathcal{J}} \cup j_0, G)$, from 4.5.15(A) one gets

$$(4.5.21) \quad \sigma_{j_0} = 1 \text{ and } j_0 \text{ is adjacent to a vertex of } \mathbf{s}(\overline{\mathcal{J}}).$$

Assume that some j_ℓ ($1 \leq \ell \leq k$) is not in $\overline{\mathcal{J}}$. Then again by $\Delta(\overline{\mathcal{J}} \cup j_\ell, G) \leq \Delta(\overline{\mathcal{J}}, G)$ and 4.5.15(A) we get that $\sigma_{j_\ell} = 1$ and j_ℓ is adjacent to $\mathbf{s}(\overline{\mathcal{J}})$. In particular, $\Delta(\overline{\mathcal{J}} \cup j_\ell, G) = \Delta(\overline{\mathcal{J}}, G)$, and we can replace $\overline{\mathcal{J}}$ by $\overline{\mathcal{J}} \cup j_\ell$. Hence, for uniform treatment, in such a situation we can always assume that

$$(4.5.22) \quad \{j_1, \dots, j_k\} \subseteq \overline{\mathcal{J}}.$$

Let \mathbf{s}_0^* be the support generated by $\{j_1, \dots, j_k\}$ via the Laufer algorithm 4.5.9; obviously, $\mathbf{s}_0^* \subseteq G_0^*$.

We will need another fact too. Let $\overline{\mathcal{J}}'$ be a subset of $\overline{\mathcal{J}}(G')$. Then

$$(4.5.23) \quad \Delta(\overline{\mathcal{J}}', G') = \Delta(\overline{\mathcal{J}}', G),$$

that is, the Δ -invariants of $\overline{\mathcal{J}}'$ in G' and in G are the same. Indeed, if $\bar{j} \notin \overline{\mathcal{J}}'$, then the identity is clear since $\overline{\mathcal{J}}'$ generates the same supports $\mathbf{s}(\overline{\mathcal{J}}', G') = \mathbf{s}(\overline{\mathcal{J}}', G)$ in G' and G . Otherwise, $\mathbf{s}(\overline{\mathcal{J}}', G)$ is the union of $\mathbf{s}(\overline{\mathcal{J}}', G')$ with the maximal element of those connected subgraph of G_0^* which are adjacent to \bar{j} and $\sigma_j = 1$ for all their vertices j .

Now, our discussion bifurcates into two cases: *whether \bar{j} is adjacent to \mathbf{s}_0^* or not.*

I. The case when \bar{j} is not adjacent to \mathbf{s}_0^* .

We start with the following general statement, valid for any $\overline{\mathcal{J}} \subseteq \overline{\mathcal{I}}$, which does not contain j_0 but it contains $\{j_1, \dots, j_k\}$. For such $\overline{\mathcal{J}}$, whenever \bar{j} is not adjacent to \mathbf{s}_0^* one has:

$$(4.5.24) \quad \Delta(\overline{\mathcal{J}}, G) = \Delta(\{j_1, \dots, j_k\}, G) + \Delta(\overline{\mathcal{J}} \cap G', G),$$

where $\overline{\mathcal{J}} \cap G'$ stands for $\overline{\mathcal{J}} \cap \mathcal{J}(G')$. For its proof run first the Laufer algorithm for the vertices $\{j_1, \dots, j_k\}$ getting \mathbf{s}_0^* , then add the remaining vertices from $\overline{\mathcal{J}} \cap G'$ and continue the algorithm.

Therefore, for any $\overline{\mathcal{J}}$ as in the assumption 4.5.20 (and with (4.5.22)) we get that $\overline{\mathcal{J}} \cap G'$ realizes $\Delta N(G')$. (Otherwise, we would be able to replace the subset $\overline{\mathcal{J}} \cap G'$ of $\overline{\mathcal{J}}$ by another

subset of $\overline{\mathcal{J}} \cap G'$ which would give larger $\Delta(\overline{\mathcal{J}} \cap G', G') = \Delta(\overline{\mathcal{J}} \cap G', G)$, cf. also with (4.5.23), which would contradict (4.5.24).) Hence, (4.5.24) combined with (4.5.23) give:

$$(4.5.25) \quad \Delta N(G) = \Delta N(G') + \Delta(\{j_1, \dots, j_k\}, G).$$

I.a. Assume that $\Delta N(G')$ can be realized by some $\overline{\mathcal{J}}'$ in G' which does not contain \overline{j} .

Then, we can apply the above statements for $\overline{\mathcal{J}} = \overline{\mathcal{J}}' \cup \{j_1, \dots, j_k\}$. Note that the Laufer algorithm runs in two independent regions cut by \overline{j} , namely in G_0^* and in $G' \setminus \overline{j}$. Hence (4.5.21) guarantees that j_0 is adjacent to \mathbf{s}_0^* .

Furthermore, if $\tilde{\mathbf{s}}(G')$ is a support answering the problem for G' , then $\tilde{\mathbf{s}} = \tilde{\mathbf{s}}(G') \cup \mathbf{s}_0^*$ is a solution for G . Note also that in this case \mathbf{s}_0^* coincides with the collection of components of $\mathbf{s}(\overline{\mathcal{J}})$ sitting in G_0^* .

I.b. Assume that all realizations of $\Delta N(G')$ by some $\overline{\mathcal{J}}'$ in G' contain \overline{j} .

Let G'_{-1} be the graph obtained from G' by replacing the decoration $\sigma_{\overline{j}}$ by $\sigma_{\overline{j}} - 1$. Then, by 4.5.15(B), we get

$$(4.5.26) \quad \Delta N(G'_{-1}) = \Delta N(G') - 1.$$

By induction, one can find a support $\tilde{\mathbf{s}}(G'_{-1})$ which solves the problem for G'_{-1} . Let \mathbf{st} be the connected (minimal) string in G_0^* adjacent to both \overline{j} and j_0 (connecting them).

If j_0 is adjacent to \mathbf{s}_0^* then $\tilde{\mathbf{s}} = \tilde{\mathbf{s}}(G'_{-1}) \cup \mathbf{s}_0^*$ is a solution for G .

Otherwise $\tilde{\mathbf{s}} = \tilde{\mathbf{s}}(G'_{-1}) \cup \mathbf{s}_0^* \cup \mathbf{st}$ is a solution for G .

II. The case when \overline{j} is adjacent to \mathbf{s}_0^* .

Note that in this case by the combinatorics of the choice of j_0 and by (4.5.21) we get that j_0 is adjacent to \mathbf{s}_0^* too.

We claim that for G'_{-1} associated with the graph G' and its vertex \overline{j} one gets

$$(4.5.27) \quad \Delta N(G) = \Delta N(G'_{-1}) + \Delta(\{j_1, \dots, j_k\}, G).$$

Moreover, $\tilde{\mathbf{s}} = \tilde{\mathbf{s}}(G'_{-1}) \cup \mathbf{s}_0^*$ is a solution for G .

4.5.28. B. Assume that for all realizations of $\Delta N(G)$ as $\Delta(\overline{\mathcal{J}}, G)$ one has $j_0 \in \overline{\mathcal{J}}$.

Replace in G the decoration σ_{j_0} by $\sigma_{j_0} - 1$, find a solution for G_{-1} , that works for G too.

This ends the proof of Proposition 4.5.2. \square

4.6. Additional properties of $\tilde{\mathbf{s}}$. Fix an integer N and $(\mathbf{i}, \overline{\mathcal{I}}) \in \overline{\mathcal{S}}_N$ as in subsection 4.5. The cube $(\mathbf{i}, \overline{\mathcal{I}})$ determines the integer $N(G) = \max_{\overline{\mathcal{J}} \subseteq \overline{\mathcal{I}}} \chi_k(x(\mathbf{i} + 1_{\overline{\mathcal{J}}}))$, cf. (4.5.6). Choose $\tilde{\mathcal{J}} \subseteq \overline{\mathcal{I}}$ which realizes this maximum: $N(G) = \chi_k(x(\mathbf{i} + 1_{\tilde{\mathcal{J}}}))$. $N(G)$ is the smallest integer N for which $(\mathbf{i}, \overline{\mathcal{I}}) \in \overline{\mathcal{S}}_N$.

Theorem 4.5.2 applied for $(\mathbf{i}, \overline{\mathcal{I}})$ and $N = N(G)$ provides a cycle $x(\mathbf{i}) + E_{\tilde{\mathbf{s}}} with $\tilde{\mathbf{s}} \subseteq \mathbf{s}(\mathbf{i}, \overline{\mathcal{I}})$ and$

$$(4.6.1) \quad \chi_k(x(\mathbf{i}) + E_{\tilde{\mathbf{s}}} + E_{\tilde{\mathcal{J}}}) \leq N(G) \text{ for any } \overline{\mathcal{J}} \subseteq \overline{\mathcal{I}}.$$

In the next paragraphs we will list some additional properties of $\tilde{\mathbf{s}}$ and $\tilde{\mathcal{J}}$.

Lemma 4.6.2. (a) $\chi_k(x(\mathbf{i}) + E_{\tilde{\mathbf{s}}} + E_{\tilde{\mathcal{J}}}) = N(G)$. In particular, the weight of the cube $(x(\mathbf{i}) + E_{\tilde{\mathbf{s}}}, \overline{\mathcal{I}})$ is $N(G)$.

(b) (i) There exists a computation sequence $\{x_n\}_{n=0}^t$ with $x_0 = x(\mathbf{i}) + E_{\tilde{\mathcal{J}}}$ and $x_t = x(\mathbf{i}) + E_{\tilde{\mathcal{J}}} + E_{\tilde{\mathbf{s}}}$ such that $\chi_k(x_{n+1}) \leq \chi_k(x_n)$ for any n .

(ii) There exists a computation sequence $\{y_n\}_{n=0}^{t'}$ with $y_0 = x(\mathbf{i}) + E_{\tilde{\mathcal{J}}} + E_{\tilde{\mathbf{s}}}$ and $y_{t'} = x(\mathbf{i}) + E_{\tilde{\mathcal{J}}} + E_{\mathbf{s}(\mathbf{i}, \overline{\mathcal{I}})}$ such that $\chi_k(x_{n+1}) \geq \chi_k(x_n)$ for any n .

(c) Using the notation $\sigma_j(J)$ from 4.5.9, one has:

$$\begin{aligned} (i) \quad & \sigma_j(\tilde{\mathbf{s}}) \geq 0 \quad \text{if } j \in \tilde{J} \\ (ii) \quad & \sigma_j(\tilde{\mathbf{s}}) \leq 0 \quad \text{if } j \notin \tilde{J}. \end{aligned}$$

Proof. Note that

$$N(G) \stackrel{(1)}{=} \chi_k(x(\mathbf{i} + 1_{\tilde{J}})) \stackrel{(2)}{\leq} \chi_k(x(\mathbf{i}) + E_{\tilde{\mathbf{s}}} + E_{\tilde{J}}) \stackrel{(3)}{\leq} N(G).$$

(1) follows from the definition of $N(G)$ and the choice of \tilde{J} , (2) from Lemma 3.2.3, and (3) from Theorem 4.5.2 applied for $N = N(G)$. This proves (a). Identity (a) together with Proposition 3.2.6(II) imply that $\tilde{\mathbf{s}} \subseteq \mathbf{s}(\mathbf{i}, \tilde{J})$. Then there exists a computation sequence connecting $x(\mathbf{i}) + E_{\tilde{J}}$ with $x(\mathbf{i}) + E_{\tilde{J}} + E_{\tilde{\mathbf{s}}}$ by 3.2.6(II), a sequence connecting $x(\mathbf{i}) + E_{\tilde{J}} + E_{\tilde{\mathbf{s}}}$ with $x(\mathbf{i}) + E_{\tilde{J}} + E_{\mathbf{s}(\mathbf{i}, \tilde{J})}$ by 3.2.6(I), and finally, from $x(\mathbf{i}) + E_{\tilde{J}} + E_{\mathbf{s}(\mathbf{i}, \tilde{J})}$ to $x(\mathbf{i}) + E_{\tilde{J}} + E_{\mathbf{s}(\mathbf{i}, \tilde{I})}$ by 3.2.6(III). This ends part (b).

Part (c) follows from (a) and equation (4.6.1) applied for $\tilde{J} \setminus \{j\}$ (case $j \in \tilde{J}$), respectively $\tilde{J} \cup \{j\}$ (case $j \notin \tilde{J}$), and from the assumption 4.1, which guarantees $(E_j, E_{\tilde{J} \setminus \{j\}}) = 0$. \square

4.6.3. Let us recall what we already proved. For any fixed $(\mathbf{i}, \tilde{I}) \in \overline{\mathcal{S}}_N$ the space $\Phi_N^*(\mathbf{i}, \tilde{I})$ is non-empty, cf. 4.5.2, and it has the homotopy type of the product (cf. 4.4.4):

$$\Phi_N^*(\mathbf{i}, \tilde{I}) = \psi(\phi_N^*(\mathbf{i}, \tilde{I})) \cap \{l^* : 0 \leq l^* - \psi(x(\mathbf{i})) \leq E_{\mathbf{s}(\mathbf{i}, \tilde{I})}\} \times (\mathbf{i}, \tilde{I}).$$

If $x \in \Phi_N^*(\mathbf{i}, \tilde{I})$ then $x - x(\mathbf{i})$ is reduced. Moreover, $\Phi_N^*(\mathbf{i}, \tilde{I})$ has in it a distinguished cube $\{\psi(x(\mathbf{i})) + E_{\tilde{\mathbf{s}}}\} \times (\mathbf{i}, \tilde{I}) = (x(\mathbf{i}) + E_{\tilde{\mathbf{s}}}, \tilde{I})$. Our goal is to construct a deformation retract from $\Phi_N^*(\mathbf{i}, \tilde{I})$ to this cube (acting in the fiber direction). This will be more complicated than the ‘standard’ retractions 4.3.1–4.3.2–4.3.3. (Note that the point $x(\mathbf{i}) + E_{\tilde{\mathbf{s}}} + E_{\tilde{J}}$ is not a χ_k -minimal point of $\Phi_N^*(\mathbf{i}, \tilde{I})$, it is maximal point in the direction $\tilde{\mathcal{J}}$ and a minimal point in the direction \mathcal{J}^* .)

To start with, we consider the connected components $\{G_\alpha\}_{\alpha \in A}$ of $\tilde{\mathbf{s}}$, and the connected components $\{C_\beta\}_{\beta \in B}$ of $\mathbf{s}(\mathbf{i}, \tilde{I}) \setminus \tilde{\mathbf{s}}$. During the contraction the supports G_α should be ‘added’ and the supports C_β should be ‘deleted’. According to this, it is performed in several steps, during one step either we add one G_α -type component, or we delete one C_β -type component. At each step the fact that which type is performed, or which G_α/C_β is manipulated is decided by a technical ‘selection procedure’. This is the subject of the next Proposition, which will be applied at any situation when the components $\{G_\alpha\}_{\alpha \in A'}$ still should be added and the components $\{C_\beta\}_{\beta \in B'}$ still should be deleted: it chooses an element of $A' \cup B'$. The technical properties associated with the corresponding cases will guarantee that the contraction stays below level N of χ_k .

Below, for any subset $\mathcal{J}' \subseteq \tilde{\mathcal{J}}$ and $i \in \mathcal{J}^*$ we write $\mathcal{J}'_i := \{j \in \mathcal{J}' : (E_i, E_j) = 1\}$.

Proposition 4.6.4. (Selection Procedure) *Fix subsets $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cup B' \neq \emptyset$. Then either there exists $\alpha \in A'$ such that*

$$(i) \quad \text{for every } i \in |G_\alpha| \text{ and every } j \in \tilde{\mathcal{J}}_i \text{ one has } \sigma_j((\tilde{\mathbf{s}} \setminus i) \cup \cup_{\beta \in B'} C_\beta) > 0$$

or, there exists $\beta \in B'$ such that

$$(ii) \quad \text{for every } i \in |C_\beta| \text{ and every } j \in \tilde{I}_i \setminus \tilde{J} \text{ one has } \sigma_j((\tilde{\mathbf{s}} \cup i) \setminus \cup_{\alpha \in A'} G_\alpha) < 0.$$

Proof. Fix some $\alpha \in A'$ and assume that it does not satisfy (i). Then there exists $i_\alpha \in |G_\alpha|$ and $j_\alpha \in \tilde{\mathcal{J}}_{i_\alpha}$ such that $\sigma_{j_\alpha}((\tilde{\mathbf{s}} \setminus i_\alpha) \cup \cup_{\beta \in B'} C_\beta) \leq 0$. Note that $\sigma_{j_\alpha}(\tilde{\mathbf{s}} \setminus i_\alpha) = \sigma_{j_\alpha}(\tilde{\mathbf{s}}) + (E_{j_\alpha}, E_{i_\alpha}) > 0$ by 4.6.2(c). These two combined prove the existence of some $\beta \in B'$ and $i_\beta \in |C_\beta|$ with $(E_{j_\alpha}, E_{i_\beta}) = 1$.

Symmetrically, if for some $\beta \in B'$ (ii) is not true, then there exists $i_\beta \in |C_\beta|$ and $j_\beta \in \bar{I}_{i_\beta} \setminus \tilde{J}$ with $\sigma_{j_\beta}((\tilde{\mathbf{s}} \cup i_\beta) \setminus \cup_{\alpha \in A'} G_\alpha) \geq 0$. Since by 4.6.2(c) we have $\sigma_{j_\beta}(\tilde{\mathbf{s}} \cup i_\beta) = \sigma_{j_\beta}(\tilde{\mathbf{s}}) - (E_{j_\beta}, E_{i_\beta}) < 0$, we get the existence of some $\alpha \in A'$ and $i_\alpha \in |G_\alpha|$ with $(E_{j_\beta}, E_{i_\alpha}) = 1$.

Now the proof runs as follows. Start with any $\alpha \in A'$. If it satisfies (i) we are done. Otherwise we get by the first paragraph a β such that G_α and C_β are connected by a length two path having the middle vertex in \tilde{J} . If this β satisfy (ii) we stop, otherwise we get by the second paragraph an α' such that C_β and $G_{\alpha'}$ are connected by a length two path whose middle vertex is not in \tilde{J} . Since the graph G has no cycles, $\alpha' \neq \alpha$. Then we continue the procedure with α' . Either it satisfies (i) or $G_{\alpha'}$ is connected with some $C_{\beta'}$ with $\beta' \neq \beta$. Continuing in this way, all the involved α indices, respectively all the β indices are pairwise distinct because of the non-existence of a cycle in the graph. Since $A' \cup B'$ is finite, the procedure must stop. \square

4.7. Contraction of $\Phi_N^*(\mathbf{i}, \bar{I})$. We will drop the symbol (\mathbf{i}, \bar{I}) from the notation $\Phi_N^*(\mathbf{i}, \bar{I})$: we write simply Φ_N^* . On the other hand, for any pair $\emptyset \subseteq \mathbf{s}_1 \subseteq \mathbf{s}_2 \subseteq \mathbf{s}(\mathbf{i}, \bar{I})$, we define

$$\Phi_N^*(\mathbf{s}_1, \mathbf{s}_2) := [\psi(\phi_N^*(\mathbf{i}, \bar{I})) \cap \{l^* : E_{\mathbf{s}_1} \leq l^* - \psi(x(\mathbf{i})) \leq E_{\mathbf{s}_2}\}] \times (\mathbf{i}, \bar{I}).$$

For example, $\Phi_N^*(\emptyset, \mathbf{s}(\mathbf{i}, \bar{I})) = \Phi_N^*$, while $\Phi_N^*(\tilde{\mathbf{s}}, \tilde{\mathbf{s}}) = \{(\psi(x(\mathbf{i})) + E_{\tilde{\mathbf{s}}})\} \times (\mathbf{i}, \bar{I})$, the cube on which we wish to contract Φ_N^* .

If the Selection Procedure chooses some $\alpha' \in A'$ then we have to construct a deformation retract

$$c_{\alpha'} : \Phi_N^*(\bigcup_{\alpha \notin A'} |G_\alpha|, \tilde{\mathbf{s}} \cup \bigcup_{\beta \in B'} |C_\beta|) \longrightarrow \Phi_N^*(\bigcup_{\alpha \notin A' \setminus \alpha'} |G_\alpha|, \tilde{\mathbf{s}} \cup \bigcup_{\beta \in B'} |C_\beta|).$$

Otherwise, if some $\beta' \in B'$ is chosen then we have to construct a deformation retract

$$c_{\beta'} : \Phi_N^*(\bigcup_{\alpha \notin A'} |G_\alpha|, \tilde{\mathbf{s}} \cup \bigcup_{\beta \in B'} |C_\beta|) \longrightarrow \Phi_N^*(\bigcup_{\alpha \notin A'} |G_\alpha|, \tilde{\mathbf{s}} \cup \bigcup_{\beta \in B' \setminus \beta'} |C_\beta|).$$

Their composition (in the selected order) provides the wished deformation retract $\Phi_N^* \rightarrow \Phi_N^*(\tilde{\mathbf{s}}, \tilde{\mathbf{s}})$.

4.7.1. The construction of $c_{\alpha'}$. Let $|G_{\alpha'}| = \{j_1, \dots, j_t\}$. By the properties of \tilde{J} , cf. 4.6.2(b), we have a computation sequence with χ_k non-increasing from $x(\mathbf{i}) + E_{\tilde{J}}$ to $x(\mathbf{i}) + E_{\tilde{J} \cup \tilde{\mathbf{s}}}$. Since the components $\{G_\alpha\}_\alpha$ do not interact, we can permute elements belonging to different components G_α , hence we may assume that the first part completed the components $\cup_{\alpha \notin A'} G_\alpha$, then we complete $G_{\alpha'}$ and the order $\{j_1, \dots, j_t\}$ is imposed by the computation sequence. Therefore, for any $1 \leq n \leq t$,

$$(4.7.2) \quad \sigma_{j_n}(\tilde{J} \cup \cup_{\alpha \notin A'} |G_\alpha| \cup \{j_1, \dots, j_{n-1}\}) \leq 0.$$

The contraction $c_{\alpha'}$ will be a composition $c_{\alpha',t} \circ \dots \circ c_{\alpha',1}$, where $c_{\alpha',n}$ corresponds to the completion of the cycles with E_{j_n} ($1 \leq n \leq t$):

$$c_{\alpha',n} : \Phi_N^*(\bigcup_{\alpha \notin A'} |G_\alpha| \cup \{j_1, \dots, j_{n-1}\}, \tilde{\mathbf{s}} \cup \bigcup_{\beta \in B'} |C_\beta|) \longrightarrow \Phi_N^*(\bigcup_{\alpha \notin A'} |G_\alpha| \cup \{j_1, \dots, j_n\}, \tilde{\mathbf{s}} \cup \bigcup_{\beta \in B'} |C_\beta|)$$

defined as follows. Write $x = x(\mathbf{i}) + E_{\tilde{J}} + l^*$ (l^* is reduced) with

$$(4.7.3) \quad \cup_{\alpha \notin A'} |G_\alpha| \cup \{j_1, \dots, j_{n-1}\} \subseteq |l^*| \subseteq \tilde{\mathbf{s}} \cup \bigcup_{\beta \in B'} |C_\beta|.$$

Then

$$c_{\alpha',n}(x) = \begin{cases} x & \text{if } j_n \in |l^*|, \\ x + E_{j_n} & \text{if } j_n \notin |l^*|. \end{cases}$$

Note that for any l^* as above with $|l^*| \not\ni j_n$, the inequality (4.7.2) implies

$$(4.7.4) \quad \sigma_{j_n}(\tilde{J} \cup |l^*|) \leq 0.$$

Fix such an l^* with $|l^*| \not\ni j_n$. Then, for *any* $\overline{J} \subseteq \overline{I}$, we have to prove

$$(4.7.5) \quad \chi_k(x(\mathbf{i}) + E_{\overline{J}} + l^* + E_{j_n}) \leq N.$$

Set $\overline{J}(l^*) := \{j \in \overline{I} : \sigma_j(|l^*|) > 0\}$. We claim that if (4.7.5) is valid for $\overline{J}(l^*)$ then it is valid for every $\overline{J} \subseteq \overline{I}$. This follows from the next identity whose second term is ≤ 0 by the definition of $\overline{J}(l^*)$.

$$(4.7.6) \quad \begin{aligned} & \chi_k(x(\mathbf{i}) + E_{\overline{J}} + l^* + E_{j_n}) - \chi_k(x(\mathbf{i}) + E_{\overline{J}(l^*)} + l^* + E_{j_n}) \\ &= \sum_{j \in \overline{J} \setminus \overline{J}(l^*)} [\sigma_j(|l^*|) - (E_j, E_{j_n})] - \sum_{j \in \overline{J}(l^*) \setminus \overline{J}} [\sigma_j(|l^*|) - (E_j, E_{j_n})]. \end{aligned}$$

On the other hand, using Selection Procedure (and its notations) we get $\tilde{J}_{j_n} \subseteq \overline{J}(l^*)$. Indeed, by the choice of α' in 4.6.4(i), for $j_n \in |G_{\alpha'}|$ and for any $j \in \tilde{J}_{j_n}$ one has $\sigma_j(\tilde{\mathbf{s}} \setminus j_n \cup \cup_{\beta \in B'} C_\beta) > 0$. Then $\sigma_j(|l^*|) > 0$ by the support condition (4.7.3). Then $\tilde{J}_{j_n} \subseteq \overline{J}(l^*)$ implies:

$$(4.7.7) \quad \sigma_{j_n}(\overline{J}(l^*) \cup |l^*|) \stackrel{(1)}{\leq} \sigma_{j_n}(\tilde{J}_{j_n} \cup |l^*|) \stackrel{(2)}{=} \sigma_{j_n}(\tilde{J} \cup |l^*|) \stackrel{(3)}{\leq} 0.$$

(1) follows from $\tilde{J}_{j_n} \subseteq \overline{J}(l^*)$, (2) from $(E_{j_n}, E_{\tilde{J}_{j_n}}) = (E_{j_n}, E_{\tilde{J}})$, and (3) from (4.7.4). Therefore,

$$\chi_k(x(\mathbf{i}) + E_{\overline{J}(l^*)} + l^* + E_{j_n}) - \chi_k(x(\mathbf{i}) + E_{\overline{J}(l^*)} + l^*) = \sigma_{j_n}(\overline{J}(l^*) \cup |l^*|) \leq 0.$$

Since $\chi_k(x(\mathbf{i}) + E_{\overline{J}(l^*)} + l^*) \leq N$ (by induction), (4.7.5) is valid for $\overline{J}(l^*)$.

4.7.8. The construction of $c_{\beta'}$. Let $|C_{\beta'}| = V_1 \cup V_2$, where $V_1 := |C_{\beta'}| \cap (\mathbf{s}(\mathbf{i}, \tilde{J}) \setminus \tilde{\mathbf{s}})$ and $V_2 := |C_{\beta'}| \cap (\mathbf{s}(\mathbf{i}, \tilde{I}) \setminus \mathbf{s}(\mathbf{i}, \tilde{J}))$. The Laufer computation sequence given by 3.2.6(I) connecting $x(\mathbf{i}) + E_{\tilde{J}} + E_{\tilde{\mathbf{s}}}$ with $x(\mathbf{i}) + E_{\tilde{J}} + E_{\mathbf{s}(\mathbf{i}, \tilde{J})}$ gives an ordering on $V_1 = \{j_1, \dots, j_{t_s}\}$ with the property

$$(4.7.9) \quad \sigma_{j_n}(\tilde{J} \cup \{j_1, \dots, j_{n-1}\}) = 0$$

for every $1 \leq n \leq t_s$. Similarly, applying 3.2.6(III) for $E_{\mathbf{s}(\mathbf{i}, \tilde{I}) \setminus \mathbf{s}(\mathbf{i}, \tilde{J})}$ we have an ordering on $V_2 = \{j_{t_s+1}, \dots, j_t\}$ such that

$$(4.7.10) \quad \sigma_{j_n}(\tilde{J} \cup \{j_1, \dots, j_{n-1}\}) \geq 0$$

for every $t_s + 1 \leq n \leq t$.

The contraction $c_{\beta'}$ will be $c_{\beta',1} \circ \dots \circ c_{\beta',t}$, where $c_{\beta',n}$ corresponds to the deletion of the cycles with E_{j_n} ($1 \leq n \leq t$), i.e.

$$\begin{aligned} c_{\beta',n} : \Phi_N^* \left(\bigcup_{\alpha \notin A'} |G_\alpha|, \tilde{\mathbf{s}} \cup \bigcup_{\beta \in B'} |C_\beta| \setminus \{j_{n+1}, \dots, j_t\} \right) \longrightarrow \\ \Phi_N^* \left(\bigcup_{\alpha \notin A'} |G_\alpha|, \tilde{\mathbf{s}} \cup \bigcup_{\beta \in B'} |C_\beta| \setminus \{j_n, \dots, j_t\} \right) \end{aligned}$$

defined in the following way. Write $x = x(\mathbf{i}) + E_{\tilde{J}} + l^*$ with

$$(4.7.11) \quad \cup_{\alpha \notin A'} |G_\alpha| \subseteq |l^*| \subseteq \tilde{\mathbf{s}} \cup \bigcup_{\beta \in B'} |C_\beta| \setminus \{j_{n+1}, \dots, j_t\},$$

then

$$c_{\beta',n}(x) = \begin{cases} x & \text{if } j_n \notin |l^*|, \\ x - E_{j_n} & \text{if } j_n \in |l^*|. \end{cases}$$

Fix such an l^* with $j_n \in |l^*|$, then we have to prove

$$(4.7.12) \quad \chi_k(x(\mathbf{i}) + E_{\tilde{J}} + l^* - E_{j_n}) \leq N$$

for any $\tilde{J} \subseteq \bar{I}$. In this case the inequalities (4.7.9) and (4.7.10) implies

$$(4.7.13) \quad \sigma_{j_n}(\tilde{J} \cup |l^*| \setminus j_n) \geq 0.$$

Here we set $\tilde{J}(l^*) := \{j \in \bar{I} : \sigma_j(|l^*|) \geq 0\}$. Then if (4.7.12) is valid for $\tilde{J}(l^*)$ then it is so for any $\tilde{J} \subseteq \bar{I}$. Indeed,

$$(4.7.14) \quad \begin{aligned} & \chi_k(x(\mathbf{i}) + E_{\tilde{J}} + l^* - E_{j_n}) - \chi_k(x(\mathbf{i}) + E_{\tilde{J}(l^*)} + l^* - E_{j_n}) \\ &= \sum_{j \in \tilde{J} \setminus \tilde{J}(l^*)} [\sigma_j(|l^*|) + (E_j, E_{j_n})] - \sum_{j \in \tilde{J}(l^*) \setminus \tilde{J}} [\sigma_j(|l^*|) + (E_j, E_{j_n})] \leq 0, \end{aligned}$$

by the definition of $\tilde{J}(l^*)$. By the selection of β' via 4.6.4(ii), for $j_n \in |C'_\beta|$ and for any $j \in \bar{I}_{j_n} \setminus \tilde{J}$ one has $\sigma_j((\tilde{\mathfrak{S}} \cup j_n) \setminus \cup_{\alpha \in A'} G_\alpha) < 0$, hence $\sigma_j(|l^*|) < 0$, in other words $\tilde{J}(l^*) \subseteq \tilde{J}$. Finally, from (4.7.13) we can deduce the inequality

$$\begin{aligned} & \chi_k(x(\mathbf{i}) + E_{\tilde{J}(l^*)} + l^* - E_{j_n}) - \chi_k(x(\mathbf{i}) + E_{\tilde{J}} + l^* - E_{j_n}) = \\ & - \sigma_{j_n}(\tilde{J}(l^*) \cup |l^*| \setminus j_n) \leq -\sigma_{j_n}(\tilde{J} \cup |l^*| \setminus j_n) \leq 0. \end{aligned}$$

5. APPLICATION. SERIES AND THE SEIBERG–WITTEN INVARIANTS.

5.1. Vanishing properties of the lattice cohomology. The first consequence of the main theorem 3.3.3 combined with Theorem 2.4.1 is the isomorphism $\mathbb{H}^*(G) = \oplus_N H^*(\bar{S}_N, \mathbb{Z})$. Since \bar{S}_N is a compact cubical subcomplex of \mathbb{R}^N , we obtain

Corollary 5.1.1. *Fix $\nu \geq 1$. If a graph G has ν bad vertices then $\mathbb{H}^q(G, k) = 0$ for any $q \geq \nu$ and $k \in \text{Char}$.*

This statement was already proved in [11] for $\nu = 1$, and in general in [18] using surgery exact sequences of lattice cohomology.

5.2. SW–invariants and series. Recall that the Seiberg–Witten invariants of the oriented 3–manifold M are rational numbers $\mathfrak{sw}_{\mathfrak{s}}(M)$ associated with the Spin^c –structures \mathfrak{s} of M . E.g., in terms of Heegaard–Floer homology

$$HF^+(M) = \oplus_{\mathfrak{s} \in \text{Spin}^c(M)} HF^+(M, \mathfrak{s}) = \oplus_{\mathfrak{s} \in \text{Spin}^c(M)} (\mathcal{T}_{d(M, \mathfrak{s})} \oplus HF_{red}^+(M, \mathfrak{s}))$$

one has

$$\mathfrak{sw}_{\mathfrak{s}}(M) = \text{rank}_{\mathbb{Z}} HF_{red, even}^+(M, \mathfrak{s}) - \text{rank}_{\mathbb{Z}} HF_{red, odd}^+(M, \mathfrak{s}) - d(M, \mathfrak{s})/2.$$

In [17] is proved that the normalized Euler characteristics of the lattice cohomology also agrees with the Seiberg–Witten invariant (note that the weight function of [17] is shifted compared with the present one; see the comment in 5.2.3(1) as well):

Theorem 5.2.1 ([17]). *Fix $k = k_{can} + 2l'$. Then*

$$-\mathfrak{sw}_{-[k]}(M(G)) - (k^2 + |\mathcal{J}|)/8 = eu(\mathbb{H}^*(G, k)),$$

where $eu(\mathbb{H}^*(G, k)) := -\mathfrak{m}_k + \sum_q (-1)^q \text{rank}_{\mathbb{Z}} \mathbb{H}^*(G, k)$.

The point we wish to emphasize is that the SW-invariant can be recovered from a combinatorial ‘zeta function’ associated with the plumbing graph as well. Indeed, consider the multivariable Taylor expansion $\sum_{l'} p_{l'} \mathbf{t}^{l'}$ of

$$(5.2.2) \quad Z(\mathbf{t}) = \prod_{j \in \mathcal{J}} (1 - \mathbf{t}^{E_j^*})^{\delta_j - 2}$$

at the origin, where δ_j is the valency of the vertex j , and for every $l' = \sum_j l_j E_j \in L'$ we write $\mathbf{t}^{l'} := \prod_{j \in \mathcal{J}} t_j^{l_j}$. This lives in $\mathbb{Z}[[L']]$, the $\mathbb{Z}[L']$ -submodule of formal power series $\mathbb{Z}[[\mathbf{t}^{\pm 1/d}]]$ in variables $\{t_j^{\pm 1/d}\}_j$, where $d = \det(-\mathfrak{J})$.

$Z(\mathbf{t})$ has a unique natural decomposition $Z(\mathbf{t}) = \sum_{h \in H} Z_h(\mathbf{t})$, where $Z_h(\mathbf{t}) = \sum_{[l'] = h} p_{l'} \mathbf{t}^{l'}$, cf. [15, 2.3]. Sometimes we also write $Z_{l'}$ for $Z_{[l']}$.

Since the Lipman cone \mathcal{S}' is $\mathbb{Z}_{\geq 0} \langle E_j^* \rangle_j$, definition (5.2.2) shows that $Z(\mathbf{t})$ is supported in \mathcal{S}' , hence $Z_{l'}(\mathbf{t})$ is supported in $(l' + L) \cap \mathcal{S}'$.

Z is also called the *combinatorial Poincaré series*. This is motivated by the following fact, for details see [3, 4, 15, 19]. We may consider the equivariant divisorial Hilbert series $\mathcal{H}(\mathbf{t})$ of a normal surface singularity $(X, 0)$ with fixed resolution graph G . The key point connecting $\mathcal{H}(\mathbf{t})$ with the topology of the link M and the graph G is introducing the series $\mathcal{P}(\mathbf{t}) = -\mathcal{H}(\mathbf{t}) \cdot \prod_j (1 - t_j^{-1}) \in \mathbb{Z}[[L']]$. Then $Z(\mathbf{t})$ is the topological candidate for $\mathcal{P}(\mathbf{t})$. They agree for several singularities, e.g. for splice quotients (see [19]), which contain all the rational, minimally elliptic or weighted homogeneous singularities. Motivated by analytic properties of \mathcal{P} (see e.g. [19]), the second author proved that $Z(\mathbf{t})$ also codifies the SW-invariants of the link M , and it is related in a subtle way with the weight function of the lattice complex.

Theorem 5.2.3 ([17]). *Fix one of the elements $l'_{[k]}$. Then the following facts hold.*

(1)

$$Z_{l'_{[k]}}(\mathbf{t}) = \mathbf{t}^{l'_{[k]}} \cdot \sum_{l \in L} \left(\sum_{I \subseteq \mathcal{J}} (-1)^{|I|+1} w(l, I) \right) \mathbf{t}^l.$$

(In [17] $w(k)$ is defined as $-(k^2 + |\mathcal{J}|)/8$ for $k \in \text{Char}$. If $k = k_r + 2l$ then $w(k) = \chi_{k_r}(l) - (k_r^2 + |\mathcal{J}|)/8$. The last constant can be neglected in the above sum since $\sum_I (-1)^{|I|} = 0$.)

(2) Assume that $l \in L$ and $l + l'_{[k]} \in -k_{\text{can}} + \text{int}(\mathcal{S}')$ and write $k = k_r + 2l$. Then

$$\sum_{\bar{l} \in L, \bar{l} \not\geq l} p_{l'_{[k]} + \bar{l}} = -\frac{(k_r + 2l)^2 + |\mathcal{J}|}{8} - \mathbf{sw}_{-[k]}(M).$$

(The sum is finite since Z is supported in $\mathbb{Z}_{\geq 0} \langle E_j^* \rangle_j$ and all the entries of E_j^* are strict positive, cf. (2.1.1).) Hence, the truncated summation from the left hand side admits a multivariable quadratic Hilbert polynomial with free term

$$-(k_r^2 + |\mathcal{J}|)/8 - \mathbf{sw}_{-[k]}(M) = eu(\mathbb{H}^*(G, k_r)).$$

Our next goal is to find the analogs of the above identities in the ‘reduced version’.

5.3. Notations. Recall that $\mathcal{J} = \overline{\mathcal{J}} \sqcup \mathcal{J}^*$, where $\overline{\mathcal{J}}$ is an index set containing all the bad vertices. Let $\phi : L \rightarrow \overline{L}$ be the projections to the $\overline{\mathcal{J}}$ -coordinates. By abuse of notation we also write $\mathbf{t} = \{t_j\}_{j \in \overline{\mathcal{J}}}$ for the monomial variables associated with \overline{L} , and $\mathbf{t}^{\mathbf{i}} = \prod_{j \in \overline{\mathcal{J}}} t_j^{i_j}$ for $\mathbf{i} = (i_1, \dots, i_\nu) \in \overline{L}$. E.g. $\mathbf{t}^{l'}|_{t_j=1, \forall j \in \mathcal{J}^*} = \mathbf{t}^{\phi(l')}$. For any $h \in H$ set

$$\overline{Z}_h(\mathbf{t}) := Z_h(\mathbf{t})|_{t_j=1 \forall j \in \mathcal{J}^*}.$$

[We warn the reader that the restricted ‘non-decomposed’ series $Z(\mathbf{t})|_{t_j=1, j \in \mathcal{J}^*}$ usually does not contain sufficient information to obtain each term $\overline{Z}_h(\mathbf{t})$ from it.]

Clearly, the support of $\overline{Z}_h(\mathbf{t})$ is contained in $\phi((l'_{[k]} + L) \cap \mathcal{S}')$, hence sits in the cone $\phi(\mathcal{S}')$. This set can be identified more precisely as follows. According to the decomposition $L' = \overline{L}' \oplus (L')^*$, the intersection matrix \mathfrak{J} has a 2×2 block structure. If (\bar{l}, l^*) are the components (in E_j -basis) of some $l' \in L'$, then $l' \in \mathcal{S}'$ if and only if

$$-\mathfrak{J} \cdot \begin{pmatrix} \bar{l} \\ l^* \end{pmatrix} = - \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \begin{pmatrix} \bar{l} \\ l^* \end{pmatrix} = \begin{pmatrix} \bar{x} \\ x^* \end{pmatrix} \geq 0.$$

This implies that $-(A - BC^{-1}B^t)\bar{l} = \bar{x} + B(-C^{-1})x^*$. Since all the entries of B and $-C^{-1}$ are positive, cf. (2.1.1), we get $-(A - BC^{-1}B^t)\bar{l} \geq 0$. Note also that the negative definiteness of \mathfrak{J} implies that $A - BC^{-1}B^t$ is negative definite as well.

Corollary 5.3.1. *Set $l'_{[k]} = (\bar{c}, c^*)$ and $\overline{\mathcal{S}}'_{l'_{[k]}} := \{\mathbf{i} \in \overline{L} : -(A - BC^{-1}B^t)(\mathbf{i} + \bar{c}) \geq 0\}$. Then $\overline{Z}_{l'_{[k]}}(\mathbf{t})$ is supported in $\overline{\mathcal{S}}'_{l'_{[k]}}$. In particular, if $\overline{Z}_{l'_{[k]}}(\mathbf{t}) = \sum_{\mathbf{i}} \bar{p}_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}$, then $\sum_{\mathbf{i}' \neq \mathbf{i}} \bar{p}_{\mathbf{i}'}$ is a finite sum.*

Remark 5.3.2. Similarly, this block decomposition language produces a precise expression for $x(\mathbf{i})$ as well. Indeed, write $x(\mathbf{i})$ as $(\mathbf{i}, x(\mathbf{i})^*) \in \overline{L} \oplus L^*$. The property $(x(\mathbf{i}) + l'_{[k]}, E_j) \leq 0$ for all $j \in \mathcal{J}^*$, from 3.1.2, reformulates as $B^t(\mathbf{i} + \bar{c}) + C(x(\mathbf{i})^* + c^*) \leq 0$. Since all the entries of C^{-1} are negative, this implies $x(\mathbf{i})^* \geq -C^{-1}B^t(\mathbf{i} + \bar{c}) - c^*$. Since $x(\mathbf{i})^* \in \mathbb{Z}^{s-\nu}$ is a minimal cycle satisfying this inequality, we get

$$(5.3.3) \quad x(\mathbf{i})^* = \lceil -C^{-1}B^t(\mathbf{i} + \bar{c}) - c^* \rceil,$$

where $\lceil \cdot \rceil$ is taken componentwise.

5.4. The next theorem provides the ‘reduced version’ of Theorem 5.2.3(1).

Theorem 5.4.1. *Let $(\overline{L}, \overline{w}[k])$ be the lattice and its compatible weight functions defined in 3.3.1. That is, $\overline{w}(\mathbf{i}, \overline{I}) = \max\{\chi_{k_r}(x(\mathbf{i} + 1_{\overline{J}})) : \overline{J} \subseteq \overline{I}\}$. Then*

$$\overline{Z}_{l'_{[k]}}(\mathbf{t}) = \mathbf{t}^{\phi(l'_{[k]})} \cdot \sum_{\mathbf{i} \in \overline{L}} \left(\sum_{\overline{I} \subseteq \overline{\mathcal{J}}} (-1)^{|\overline{I}|+1} \overline{w}(\mathbf{i}, \overline{I}) \right) \mathbf{t}^{\mathbf{i}}.$$

Proof. For simplicity we abbreviate k_r into k and $\overline{w}[k]$ into \overline{w} . By 5.2.3(1) we get

$$\overline{Z}_{l'_{[k]}}(\mathbf{t}) = \mathbf{t}^{\phi(l'_{[k]})} \cdot \sum_{\mathbf{i} \in \overline{L}} \sum_{\overline{I} \subseteq \overline{\mathcal{J}}} (-1)^{|\overline{I}|+1} \left(\sum_{l^* \in L^*} \sum_{I^* \subseteq \mathcal{J}^*} (-1)^{|I^*|} w(x(\mathbf{i}) + l^*, \overline{I} \cup I^*) \right) \mathbf{t}^{\mathbf{i}},$$

where $L^* \subset L$ is the sublattice of \mathcal{J}^* -coordinates. For a fixed \mathbf{i} and $\overline{I} \subseteq \overline{\mathcal{J}}$, denote the coefficient in the last bracket by $T = T(\mathbf{i}, \overline{I})$. Then we have to show that $T = \overline{w}(\mathbf{i}, \overline{I})$.

We define a weighted lattice (L^*, w^*) as follows: the weight of a cube (l^*, I^*) in L^* is $w^*(l^*, I^*) := w(x(\mathbf{i}) + l^*, \overline{I} \cup I^*)$ (hence it depends on $(\mathbf{i}, \overline{I})$). This is a compatible weight function on L^* since w is so, moreover $T = \sum_{l^* \in L^*} \sum_{I^* \subseteq \mathcal{J}^*} (-1)^{|I^*|} w^*(l^*, I^*)$.

Note also that for any fixed \mathbf{i} the possible $l^* \in L^*$ for which $(\mathbf{i}, l^*) \in \mathcal{S}'$ is finite (use (2.1.1)), hence the sum in T is finite, and, in fact, we can find a rectangle $R^* = R^*(l_1^*, l_2^*) = \{l^* \in L^* : l_1^* \leq l^* \leq l_2^*\}$ for some l_1^* and l_2^* such that

$$T = \sum_{l^* \in R^*} \sum_{I^* \subseteq \mathcal{J}^*} (-1)^{|I^*|} w^*(l^*, I^*) \quad \text{and} \quad \mathbb{H}^*(L^*, w^*) = \mathbb{H}^*(R^*, w^*)$$

(see also subsections 4.3 and 4.4.) Using the result and methods of [17, Theorem 2.3.7], for the counting function $\mathcal{M}(t) := \sum_{l^* \in R^*} \sum_{I^* \subseteq \mathcal{J}^*} (-1)^{|I^*|} t^{w^*(l^*, I^*)}$ we get

$$\lim_{t \rightarrow 1} \frac{\mathcal{M}(t) - t^{\min(w^*|_{R^*})}}{1-t} = \sum_{q \geq 0} (-1)^q \text{rank}_{\mathbb{Z}}(\mathbb{H}_{red}^q(R^*, w^*)).$$

The Reduction Theorem 3.3.3 and its proof says that (L^*, w^*) has vanishing reduced cohomology, in particular $\mathbb{H}_{red}^q(R^*, w^*) = 0$ for any $q \geq 0$. Hence

$$T = \left. \frac{d\mathcal{M}(t)}{dt} \right|_{t=1} = \min(w^*|_{R^*}) = \min_{l^* \in L^*} \{w(x(\mathbf{i}) + l^*, \bar{I})\} = \min_{l^* \in L^*} \max_{\bar{J} \subseteq \bar{I}} \{\chi_k(x(\mathbf{i}) + l^* + E_{\bar{J}})\}.$$

By Lemma 3.2.3 $\chi_k(x(\mathbf{i}) + l^* + E_{\bar{J}}) \geq \chi_k(x(\mathbf{i} + 1_{\bar{J}}))$, hence

$$(5.4.2) \quad \max_{\bar{J} \subseteq \bar{I}} \chi_k(x(\mathbf{i}) + l^* + E_{\bar{J}}) \geq \max_{\bar{J} \subseteq \bar{I}} \chi_k(x(\mathbf{i} + 1_{\bar{J}})) = \bar{w}(\mathbf{i}, \bar{I}).$$

But, by Lemma 4.6.2(a) (for notations see also 4.6), the minimum over l^* of the left hand side is realized for $l^* = E_{\bar{s}}$ with equality in (5.4.2), hence $T = \bar{w}(\mathbf{i}, \bar{I})$. \square

5.5. In order to prove the ‘reduced version’ of Theorem 5.2.3(2) we need the analogue of Lemmas 4.3.2 and 4.4.3 in (\bar{L}, \bar{w}) .

Lemma 5.5.1. (a) *There exists $\mathbf{i}_0 \in \bar{\mathcal{S}}'_{l'_{[k]}}$ such that for any $\mathbf{i} \in (\mathbf{i}_0 + \bar{\mathcal{S}}'_{l'_{[k]}}) \cap \bar{L}$ we have $\bar{w}(\mathbf{i} + 1_j) > \bar{w}(\mathbf{i})$ and $\mathbf{i} + 1_j \in \mathbf{i}_0 + \bar{\mathcal{S}}'_{l'_{[k]}}$ for every $j \in \bar{\mathcal{J}}$.*

(b) *There exists some $\mathbf{i} \in \mathbf{i}_0 + \bar{\mathcal{S}}'_{l'_{[k]}}$ and a sequence $\{z_n\}_{n \geq 0} \in \mathbf{i}_0 + \bar{\mathcal{S}}'_{l'_{[k]}}$ such that*

- (i) $z_0 = \mathbf{i}$, $z_{n+1} = z_n + E_{j(n)}$ ($j(n) \in \bar{\mathcal{J}}$) and $\lim_{n \rightarrow \infty} (z_n)_j = \infty$ for all $j \in \bar{\mathcal{J}}$;
- (ii) *for any $n \geq 0$ and $0 \leq z'_n \leq z_n$ with the same $j(n)$ -th coefficients, one has*

$$\bar{w}(z'_n + E_{j(n)}) > \bar{w}(z'_n).$$

Proof. (a) follows from the negative definiteness of $A - BC^{-1}B^t$. Indeed, by 3.2.5 the needed property for \mathbf{i} is equivalent to $(x(\mathbf{i}) + l'_{[k]}, E_j) \leq 0$ for all $j \in \bar{\mathcal{J}}$. This reads as $A(\mathbf{i} + \bar{c}) + B(x(\mathbf{i})^* + c^*) \leq 0$, which via (5.3.3) transforms into $A(\mathbf{i} + \bar{c}) + B[-C^{-1}B^t(\mathbf{i} + \bar{c}) - c^*] + Bc^* \leq 0$. This is implied e.g. by $-(A - BC^{-1}B^t)(\mathbf{i} + \bar{c}) \geq B\mathbf{1}$ which definitely has solutions.

(b) Note that for any \mathbf{i} one has $\bar{w}(\mathbf{i} + 1_j) = \chi_k(x(\mathbf{i} + E_j))$, cf. Proposition 3.2.5, hence $\bar{w}(\mathbf{i} + 1_j) > \bar{w}(\mathbf{i})$ is equivalent with $(E_j, x(\mathbf{i}) + l'_{[k]}) \leq 0$. The same is true for any \mathbf{i}' with $0 \leq \mathbf{i}' \leq \mathbf{i}$. Since $\mathbf{i}' \leq \mathbf{i}$ implies $x(\mathbf{i}') \leq x(\mathbf{i})$, cf. 3.1.2(c)(iii), if \mathbf{i}' and \mathbf{i} have the same j -th coefficient, then $(E_j, x(\mathbf{i}') + l'_{[k]}) \leq (E_j, x(\mathbf{i}) + l'_{[k]})$. Hence $\bar{w}(\mathbf{i} + 1_j) > \bar{w}(\mathbf{i})$ implies $\bar{w}(\mathbf{i}' + 1_j) > \bar{w}(\mathbf{i}')$. Hence any increasing sequence inside of $\mathbf{i}_0 + \bar{\mathcal{S}}'_{l'_{[k]}}$ works. \square

Similarly as in subsection 4.4, one obtains (via a retract compatible with χ_k)

$$(5.5.2) \quad \mathbb{H}^*(\bar{L}, \bar{w}) \cong \mathbb{H}^*(R(0, \mathbf{i}), \bar{w}),$$

where $R(0, \mathbf{i}) = \{\mathbf{i}' \in \bar{L} : 0 \leq \mathbf{i}' \leq \mathbf{i}\}$, and \mathbf{i} is as in 5.5.1(b). In particular, if we set

$$\mathcal{E}(R(0, \mathbf{i})) := \sum_{(\mathbf{i}', \bar{I}) \subseteq R(0, \mathbf{i})} (-1)^{|\bar{I}|+1} \bar{w}(\mathbf{i}', \bar{I})$$

(sum over all the cubes of $R(0, \mathbf{i})$), then [17, Theorem 2.3.7] ensures that

$$(5.5.3) \quad \mathcal{E}(R(0, \mathbf{i})) = eu(\mathbb{H}^*(R(0, \mathbf{i}), \bar{w})),$$

which by the above discussions equals $eu(\mathbb{H}^*(\bar{L}, \bar{w})) = eu(\mathbb{H}^*(G, k_r))$.

With all these preparations, the analogue of Theorem 5.2.3(2) follows by identical proof as the original one (cf. [17, Theorem 3.1.1]).

Theorem 5.5.4. *Set $\bar{Z}_{l'_{[k]}} = \mathbf{t}^{\phi(l'_{[k]})} \cdot \sum_{\mathbf{i}} \bar{p}_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}$. For every $\mathbf{i} \in (\mathbf{i}_0 + \bar{\mathcal{S}}'_{l'_{[k]}}) \cap \bar{L}$ as in Lemma 5.5.1*

$$\sum_{\mathbf{i}' \neq \mathbf{i}} \bar{p}_{\mathbf{i}'} = \bar{w}(\mathbf{i}) - (k_r^2 + |\mathcal{J}|)/8 - \mathbf{sw}_{-[k]}(M).$$

Proof. We choose a computation sequence $\{z_n\}_{n \geq 0}$ as in 5.5.1 for the specified $\mathbf{i} \in \bar{L}$. Then set $R' := \{\mathbf{i}' \in \bar{L} : \mathbf{i}' \geq 0 \text{ and } \exists j \in \bar{\mathcal{J}} \text{ with } (\mathbf{i}' - \mathbf{i})_j \leq 0\}$. R' is not finite, but $R' \cap \bar{\mathcal{S}}'_{l'_{[k]}}$ is a finite set. Fix \tilde{n} so that $R' \cap \bar{\mathcal{S}}'_{l'_{[k]}} \subseteq R(0, z_{\tilde{n}})$, and define $R'(\tilde{n}) := R' \cap R(0, z_{\tilde{n}})$, $\partial_1 R'(\tilde{n}) := R' \cap R(\mathbf{i}, z_{\tilde{n}})$, and

$$\partial_2 R'(\tilde{n}) := \{\mathbf{i}' \in R'(\tilde{n}) : \exists j \in \bar{\mathcal{J}} \text{ with } (\mathbf{i}' - z_{\tilde{n}})_j = 0\}.$$

Then by Theorem 5.4.1 we have

$$\sum_{\mathbf{i}' \neq \mathbf{i}} \bar{p}_{\mathbf{i}'} = \mathcal{E}(R'(\tilde{n})) - \mathcal{E}(\partial_1 R'(\tilde{n}) \cup \partial_2 R'(\tilde{n})).$$

First, notice that we may find \tilde{n} in such a way, that if we choose a sequence $\{\mathbf{i}_m\}_{m=0}^t$ from $\mathbf{i}_0 = 0$ to $\mathbf{i}_t = \mathbf{i}$ with $\mathbf{i}_{m+1} = \mathbf{i}_m + 1_{j(m)}$, we have the following property:

for every $\mathbf{i}' \in \partial_2 R'(\tilde{n})$ with $\mathbf{i}' \geq \mathbf{i}_m$ and $(\mathbf{i}')_{j(m)} = (\mathbf{i}_m)_{j(m)}$ one has $\bar{w}(\mathbf{i}' + 1_{j(m)}) \leq \bar{w}(\mathbf{i}')$.

Indeed, $(x(\mathbf{i}') + l'_{[k]}, E_{j(m)})$ is increasing in \mathbf{i}' with fixed $j(m)$ -th coefficient, hence for a suitable big \tilde{n} the above property is guaranteed. (Any $\mathbf{i}' \in \partial_2 R'(\tilde{n})$ has some ‘large’ coordinate entries corresponding to coordinates j when $(\mathbf{i}' - z_{\tilde{n}})_j = 0$, and ‘small’ entries corresponding to coordinates j when $(\mathbf{i}' - \mathbf{i}_m)_j = 0$. Hence, when we wish to increase the small entries, the positivity of the quantities $(x(\mathbf{i}') + l'_{[k]}, E_{j(m)})$ is guaranteed by the presence of ‘large’ entries.)

Therefore, using the sequence $\{\mathbf{i}_m\}$ and 4.3.3, there exists a contraction of $\partial_2 R'(\tilde{n})$ to $\partial_1 R'(\tilde{n}) \cap \partial_2 R'(\tilde{n})$ along which \bar{w} is non-increasing. Then $\mathcal{E}(\partial_2 R'(\tilde{n})) = \mathcal{E}(\partial_1 R'(\tilde{n}) \cap \partial_2 R'(\tilde{n}))$ by 4.4 and (5.5.3), hence $\mathcal{E}(\partial_1 R'(\tilde{n}) \cup \partial_2 R'(\tilde{n})) = \mathcal{E}(\partial_1 R'(\tilde{n}))$.

Next, we claim that $\mathcal{E}(R'(\tilde{n})) = \mathcal{E}(R(0, \mathbf{i}))$ (which equals $eu(\mathbb{H}^*(G, k_r))$ by (5.5.3)). Indeed, using induction on the sequence $\{z_n\}_{0 \leq n < \tilde{n}}$, it is enough to show that $\mathcal{E}(R'(n)) = \mathcal{E}(R'(n+1))$. This can be seen using the second part of 5.5.1, which shows that for all \bar{I} containing $j(n)$ and each $(\mathbf{i}', \bar{I}) \in R'(n+1) \setminus R'(n)$ we have

$$\bar{w}(\mathbf{i}', \bar{I}) = \bar{w}(\mathbf{i}' + 1_{j(n)}, \bar{I} \setminus j(n)).$$

This ensures a combinatorial cancelation in the sum $\mathcal{E}(R'(n+1))$, or an isomorphism in the corresponding lattice cohomologies, which gives the expected equality.

With the same procedure applying to $\partial_1 R'(\tilde{n})$ we deduce the equality $\mathcal{E}(\partial_1 R'(\tilde{n})) = \mathcal{E}(\partial_1 R'(0)) = -\bar{w}(\mathbf{i})$. Hence the identity follows. \square

5.6. Final Remark. The reduction of ‘graded roots’. Associated with a set of compatible weight functions (L, w) , the second author in [11, 13] introduced a set of graded roots (one root for each k_r), as a refinement of the 0-th order lattice cohomology $\mathbb{H}^0(L, w)$. Without saying more about their definition or properties, we note that the very same proof of the reduction theorem provides an identification of the two graded roots associated with $(L, w) = (L, \chi_{k_r})$ and $(\bar{L}, \bar{w}[k])$ as well (similarly as in Theorem 3.3.3 for \mathbb{H}^*).

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